APPLICATION AND IMPLEMENTATION OF INCORPORATING LOCAL BOUNDARY CONDITIONS INTO NONLOCAL PROBLEMS

BURAK AKSOYLU, HORST REINHARD BEYER AND FATIH CELIKER

Department of Mathematics, TOBB University of Economics and Technology, Ankara, 06560, Turkey & Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202, USA.

Department of Mathematics, TOBB University of Economics and Technology, Ankara, 06560, Turkey & Universidad Politécnica de Uruapan, Carretera Carapan, 60120 Uruapan Michoacán, México, & Theoretical Astrophysics, IAAT, Eberhard Karls University of Tübingen, Tübingen 72076, Germany.

Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202, USA.

Abstract. We study nonlocal equations from the area of peridynamics, an instance of nonlocal wave equation, and nonlocal diffusion on bounded domains whose governing equations contain a convolution operator based on integrals. We generalize the notion of convolution in order to accommodate local boundary conditions. On a bounded domain, the classical operator with local boundary conditions has a purely discrete spectrum, and hence, provides a Hilbert basis. We define an abstract convolution operator using this Hilbert basis, thereby, automatically satisfying local boundary conditions. The main goal in this paper is twofold: apply the concept of abstract convolution operator to nonlocal problems and carry out a numerical study of the resulting operators. We study the corresponding initial value problems with prominent boundary conditions such as periodic, antiperiodic, Neumann, and Dirichlet. In order to connect to the standard convolution, we give an integral representation of the abstract convolution operator. For discretization, we employ a weak formulation based on a Galerkin projection and use piecewise polynomials on each element which allows discontinuities of the approximate solution at the element borders. We study convergence order of solutions with respect to polynomial order and observe optimal convergence. We depict the solutions for each boundary condition.

Keywords: Nonlocal wave equation, nonlocal operator, peridynamics, boundary conditions, Galerkin projection method, operator theory.

Mathematics Subject Classification (2000): 35L05, 74B99.

1. Introduction

We study a class of nonlocal wave equations. The driving application is peridynamics (PD) whose equation of motion corresponds exactly to the nonlocal wave equation under consideration. The same operator is also employed in nonlocal diffusion [10, 13, 29]. Similar classes of operators are used in numerous applications such as coagulation, image processing, particle systems, phase transition, population models.

PD is a nonlocal extension of continuum mechanics developed by Silling [30], is capable of quantitatively predicting the dynamics of propagating cracks, including bifurcation. Its effectiveness has been established in

E-mail address: burak@wayne.edu, horstbeyer@gmail.com, celiker@wayne.edu.

Date: February 10, 2017.

Burak Aksoylu was supported in part by the National Science Foundation DMS 1016190 grant, European Commission Marie Curie Career Integration 293978 grant, and Scientific and Technological Research Council of Turkey (TÜBİTAK) MFAG 115F473 grant. Research visit of Horst R. Beyer was supported in part by the TÜBİTAK 2221 Fellowship for Visiting Scientist Program. Sabbatical visit of Fatih Celiker was supported in part by the TÜBİTAK 2221 Fellowship for Scientist on Sabbatical Leave Program.

Fatih Celiker was supported in part by the National Science Foundation DMS 1115280 grant.

sophisticated applications such as Kalthoff-Winkler experiments of the fracture of a steel plate with notches [22, 32], fracture and failure of composites, nanofiber networks, and polycrystal fracture [23, 28, 34, 33]. Also see the review and news articles [13, 14, 16, 31] for a comprehensive discussion, and the recent book [25]. In addition, we witness a major effort to meet the need for mathematical theory for PD applications and related nonlocal problems addressing, for instance, conditioning analysis, domain decomposition and variational theory [7, 8, 9], discretization [9, 18, 35], nonlinear PD [19, 24], convergence of solutions [15, 17, 26, 27, 37]. Since PD is a nonlocal theory, one might expect only the appearance of nonlocal boundary conditions (BC). Indeed, so far the concept of local BC does not apply to PD. Instead, external forces must be supplied through the loading force density b [30]. On the other hand, we demonstrate that the anticipation that local BC are incompatible with nonlocal operators is not quite correct.

In the unbounded domain case, in [11], we discovered that the governing nonlocal operator is a function of a multiple of the classical governing operator. Therefore, for the bounded domain case, it was natural to define the governing operator as a function of the corresponding classical operator. This opened a gateway to incorporate local BC to nonlocal theories, which is the main theme of our foundation paper [1]. In [3], we constructed novel governing operators in 1D that agree with the original bond-based PD operator in the bulk of the domain and simultaneously enforce local Neumann and Dirichlet BC. In [6], we extended the novel governing operators in 1D to arbitrary dimension. In [2], we studied other related governing operators. In [5], we give an overview of local BC in general nonlocal problems.

A multiple of the Laplace operator with appropriate BC is chosen as the classical operator, which we denote by $A_{\mathtt{BC}}$. In the bounded domain case, the spectrum of the Laplace operator with classical BC such as periodic, antiperiodic, Neumann, and Dirichlet, is purely discrete. Furthermore, we can explicitly calculate the eigenfunctions $e_k^{\mathtt{BC}}$ corresponding to each BC and the subscript signifies the BC used; $\mathtt{BC} \in \{\mathtt{p}, \mathtt{a}, \mathtt{N}, \mathtt{D}\}$ where $\mathtt{p}, \mathtt{a}, \mathtt{N}$, and $\mathtt{D}$ stand for periodic, antiperiodic, Neumann, Dirichlet, respectively. Let $\Omega := (-1, 1)$ be the domain of interest throughout the paper. These eigenfunctions form a Hilbert (complete and orthonormal) basis for $L^2_{\mathbb{C}}(\Omega)$ through which the abstract convolution can be defined as follows

$$\mathcal{C} *_{\mathtt{BC}} u := \sum_k \langle e_k^{\mathtt{BC}} | C \rangle \, \langle e_k^{\mathtt{BC}} | u \rangle \, e_k^{\mathtt{BC}}, \tag{1.1}$$

where $\langle \cdot | \cdot \rangle$ denotes the inner product in $L^2_{\mathbb{C}}(\Omega)$ and is defined by

$$\langle e_k^{\mathtt{BC}} | u \rangle := \int_{-1}^{1} \left(e_k^{\mathtt{BC}}\right)^*(y) u(y) dy.$$

Throughout the paper, we assume that $C \in L^2(\Omega)$ is an even function. Namely,

$$C(-y) = C(y). \tag{1.2}$$

Inspired by the governing equation on the unbounded domain, we define the nonlocal wave equation

$$u_{tt}(x,t) + f_{\mathtt{BC}}(A_{\mathtt{BC}}) u(x,t) = 0, \quad x \in \Omega, \; t > 0, \tag{1.3}$$

where $f_{\mathtt{BC}} : \sigma(A_{\mathtt{BC}}) \to \mathbb{R}$ is a bounded function and $\sigma(A_{\mathtt{BC}})$ denotes the spectrum of $A_{\mathtt{BC}}$. The convolution in (1.1) is used in the governing operator $c - \mathcal{C} *_{\mathtt{BC}}$ where c is a suitable constant [1] and *regulating function* $f_{\mathtt{BC}}$ is defined as

$$f_{\mathtt{BC}}(A_{\mathtt{BC}}) := c - \mathcal{C} *_{\mathtt{BC}} \; .$$

The class of nonlocal problems we consider has a governing equation that involves a convolution operator. Both 1D bond based PD and nonlocal diffusion fall into this class. This type of governing equation with prominent BC such as periodic, antiperiodic, Dirichlet, and Neumann are all instances of regular Sturm-Liouville problems. For these problems, all BC leading to self-adjoint operators are known [36, Thm. 13.14]. If needed, all associated BC can be considered. All regular Sturm-Liouville operators are known to have a purely discrete spectrum, in particular, there is a Hilbert basis of eigenfunctions. There are a number of standard problems in higher dimensions that can be reduced to regular Sturm-Liouville problems on bounded domains. Also, generically, a differential operator with regular coefficients on $\mathbb{R}^n$ has a purely discrete spectrum, providing an eigenbasis of the underlying space. Since the essential ingredient is a self-adjoint operator with a purely discrete spectrum, hence, our approach can easily cover higher spatial dimensions.

[1] In the PD context, the choice of c in practice is $\int_\Omega C(y)dy$ with $C \geq 0$.

$$\begin{tikzcd}
 & A_{0,\mathtt{p}} \arrow[r] & A_{\mathtt{p}} \\
 & A_{0,\mathtt{a}} \arrow[r] & A_{\mathtt{a}} \\
A_0 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & A_{0,\mathtt{N}} \arrow[r] & A_{\mathtt{N}} \\
 & A_{0,\mathtt{D}} \arrow[r] & A_{\mathtt{D}}
\end{tikzcd}$$

FIGURE 2.1. We extend the minimal operator A_0, specifying boundary conditions such as periodic, antiperiodic, Neumann, and Dirichlet boundary conditions, to an essentially self-adjoint operator $A_{0,\mathtt{p}}, A_{0,\mathtt{a}}, A_{0,\mathtt{N}}, A_{0,\mathtt{D}}$, respectively. Finally, we arrive at self-adjoint operators $A_{\mathtt{p}}, A_{\mathtt{a}}, A_{\mathtt{N}}, A_{\mathtt{D}}$ by taking the closure of $A_{0,\mathtt{p}}, A_{0,\mathtt{a}}, A_{0,\mathtt{N}}, A_{0,\mathtt{D}}$, respectively.

The main goal in this paper is twofold: apply the concept of abstract convolution operator to nonlocal problems such as the nonlocal wave equation given in (1.3) and carry out a numerical study of the resulting operators. The choice of the Hilbert basis provides a flexibility in the construction of the abstract convolution operator. We make this construction concrete by choosing the basis to be the eigenbasis of the classical operator with prominent local BC indicated above. This is precisely the mechanism we use to incorporate local BC into nonlocal problems. The theoretical aspects and foundations of this construction process are discussed in our foundation paper [1].

The rest of the article is structured as follows. In Sec. 2, we define the classical operators with the prescribed BC, domains of the operators, and the corresponding eigenpairs. In Sec. 3, we apply the theoretical results from the foundation paper [1] to prominent BC such as periodic, antiperiodic, Neumann, and Dirichlet BC. We identify the abstract convolution operator (1.1) as *canonical*. We obtain integral representations of canonical convolution operators because they are more convenient for implementation. In the case of periodic and antiperiodic BC, integral representations of the canonical convolutions are relatively direct to establish. On the other hand, for Neumann boundary condition, this representation is considerably more involved, requiring arguments related to half-wave symmetry of functions. For Dirichlet BC, we give representation in terms limits of integral convolutions.

In Sec. 4, for Neumann and Dirichlet conditions, we give alternative governing operators that are structurally simpler than canonical operators. These *simple* convolutions that are derived from certain combinations of the periodic and antiperiodic extensions of the micromodulus function. They are used in numerical experiments.

In Sec. 5, we present a comprehensive numerical treatment of the nonlocal wave equation. We have two goals in numerical experiments. First, we want to demonstrate that discontinuities of the initial data remain stationary for $t \in \mathbb{R}$. Second, solutions satisfy the BC also for $t \in \mathbb{R}$. In order to show that the two goals are accomplished, we choose discontinuous initial data and run experiments showing wave evolutions for all of the considered BC; periodic, antiperiodic, Neumann, and Dirichlet. Furthermore, by choosing continuous initial data, we draw parallels between the local and nonlocal wave equations for Neumann and Dirichlet BC. We conclude in Sec. 6.

2. Operator Definition and the Corresponding Eigenpairs

We define the minimal operator $A_0 : C_0^2(\Omega, \mathbb{C}) \to L^2_{\mathbb{C}}(\Omega)$ by

$$A_0 u := -a_0\, u'',$$

where a_0 is a suitable real number and $u \in C_0^2(\Omega, \mathbb{C})$. The operator A_0 is densely defined, linear, and symmetric, but not essentially self-adjoint. We give self-adjoint extensions of A_0 by the closure of essentially self-adjoint operators. The extension process is depicted in Fig. 2.1.

Based on this construction, we define the minimal operators of interest on $\Omega := (-1, 1)$ as follows

$$A_{0,\mathtt{BC}} : D(A_{0,\mathtt{BC}}) \to L^2_{\mathbb{C}}(\Omega),$$

whose domains and definitions are given, respectively,

$$\begin{aligned}
D(A_{0,\mathtt{p}}) &:= \big\{u \in C^2(\bar{\Omega},\mathbb{C}) : \lim_{x\to-1} u(x) = \lim_{x\to 1} u(x),\ \lim_{x\to-1} u'(x) = \lim_{x\to 1} u'(x)\big\},\\
D(A_{0,\mathtt{a}}) &:= \big\{u \in C^2(\bar{\Omega},\mathbb{C}) : \lim_{x\to-1} u(x) = -\lim_{x\to 1} u(x),\ \lim_{x\to-1} u'(x) = -\lim_{x\to 1} u'(x)\big\},\\
D(A_{0,\mathtt{N}}) &:= \big\{u \in C^2(\bar{\Omega},\mathbb{C}) : \lim_{x\to-1} u'(x) = \lim_{x\to 1} u'(x) = 0\big\},\\
D(A_{0,\mathtt{D}}) &:= \big\{u \in C^2(\bar{\Omega},\mathbb{C}) : \lim_{x\to-1} u(x) = \lim_{x\to 1} u(x) = 0\big\},
\end{aligned}$$

where the governing operator is defined as

$$A_{0,\mathtt{BC}}u := -a_{0,\mathtt{BC}}\, u'', \quad u \in D(A_{0,\mathtt{BC}}),$$

with

$$a_{0,\mathtt{p}} = a_{0,\mathtt{a}} = \frac{1}{\pi^2}, \quad a_{0,\mathtt{N}} = a_{0,\mathtt{D}} = \frac{4}{\pi^2}.$$

The space $C^2(\bar{\Omega},\mathbb{C})$ consists of the restrictions of the elements of $C^2(J,\mathbb{C})$ to Ω, where J runs through all open intervals of $\mathbb{R}$ containing $\bar{\Omega}$. Note that $C^2(\bar{\Omega},\mathbb{C})$ is a *dense subspace* of $X := L^2_{\mathbb{C}}(\Omega)$ and $A_{0,\mathtt{BC}}$ is a densely-defined, linear, and symmetric operator. We also note that $A_{0,\mathtt{BC}}$ is a special case of a regular Sturm-Liouville operator. In particular, it is essentially self-adjoint. The closure of $A_{0,\mathtt{BC}}$ is $A_{\mathtt{BC}}$ and is given by

$$A_{\mathtt{BC}}u = -a_{0,\mathtt{BC}}\, u'',$$

where $'$ denotes the weak derivative. The function u is a restriction to Ω, for $\mathtt{BC} = \mathtt{p}, \mathtt{a}$, of an periodic and an antiperiodic element of $W^2(\mathbb{R},\mathbb{C})$, and, for $\mathtt{BC} = \mathtt{N}, \mathtt{D}$, of element of $W^2_0(\Omega,\mathbb{C})$, respectively.

2.1. **Spectral Information, the Associated Hilbert Basis, and Compactness.** Each operator has a purely discrete spectrum consisting of the following eigenvalues

$$\begin{aligned}
\sigma(A_{\mathtt{p}}) &= \big\{k^2 : k \in \mathbb{N}\big\}, \quad \sigma(A_{\mathtt{a}}) = \big\{(k+\frac{1}{2})^2 : k \in \mathbb{N}\big\},\\
\sigma(A_{\mathtt{N}}) &= \big\{k^2 : k \in \mathbb{N}\big\}, \quad \sigma(A_{\mathtt{D}}) = \big\{k^2 : k \in \mathbb{N}^*\big\}.
\end{aligned}$$

The corresponding normalized eigenfunctions are as follows

$$\begin{aligned}
e^{\mathtt{p}}_k(x) &:= \frac{1}{\sqrt{2}}\, e^{i\pi k x}, \quad k \in \mathbb{N}, & e^{\mathtt{a}}_k(x) &:= \frac{1}{\sqrt{2}}\, e^{i\pi(k+\frac{1}{2})x}, \quad k \in \mathbb{N},\\
e^{\mathtt{N}}_k(x) &:= \begin{cases} \frac{1}{\sqrt{2}}, & k = 0,\\ \cos\big(\frac{k\pi}{2}(x+1)\big), & k \in \mathbb{N}^*, \end{cases} & e^{\mathtt{D}}_k(x) &:= \sin\big(\frac{k\pi}{2}(x+1)\big), \quad k \in \mathbb{N}^*.
\end{aligned}$$

Then, we have the following.

- **Periodic:** $(e^{\mathtt{p}}_k)_{k\in\mathbb{Z}}$ is a Hilbert basis of $L^2_{\mathbb{C}}(\Omega)$, 0 is a simple eigenvalue and for every $k \in \mathbb{N}^*$, k^2 is an eigenvalue of geometric multiplicity 2, with corresponding linearly independent eigenfunctions $e^{\mathtt{p}}_k, e^{\mathtt{p}}_{-k}$.
- **Antiperiodic:** $(e^{\mathtt{a}}_k)_{k\in\mathbb{Z}}$ is a Hilbert basis of $L^2_{\mathbb{C}}(\Omega)$, and for every $k \in \mathbb{N}$, $(k+(1/2))^2$ is an eigenvalue of geometric multiplicity 2, with corresponding linearly independent eigenfunctions $e^{\mathtt{a}}_k, e^{\mathtt{a}}_{-k-1}$.
- **Neumann:** $e^{\mathtt{N}}_0, e^{\mathtt{N}}_1, \dots$ is a Hilbert basis of $L^2_{\mathbb{C}}(\Omega)$. Also note for $k \in \mathbb{N}^*$ that

$$e^{\mathtt{N}}_{2k}(x) = (-1)^k \cos(k\pi x), \quad e^{\mathtt{N}}_{2k-1}(x) = (-1)^k \sin\big(\pi\big(k-\frac{1}{2}\big)x\big).$$

- **Dirichlet:** $e^{\mathtt{D}}_1, e^{\mathtt{D}}_2, \dots$ is a Hilbert basis of $L^2_{\mathbb{C}}(\Omega)$. Also note for $k \in \mathbb{N}^*$ and $l \in \mathbb{N}$ that

$$e^{\mathtt{D}}_{2k}(x) = (-1)^k \sin(k\pi x), \quad e^{\mathtt{D}}_{2l+1}(x) = (-1)^l \cos\big(\pi\big(l+\frac{1}{2}\big)x\big).$$

The following result outlines under what conditions $f_{\mathtt{BC}}(A_{\mathtt{BC}})$ becomes a compact operator.

Lemma 2.1. (Compactness) *Let $B(\sigma(A),\mathbb{C})$ denote the space of complex valued bounded functions on $\sigma(A)$. Then, for*

$$
\begin{aligned}
f_{\mathtt{p}}(A_{\mathtt{p}}) &\in B(\sigma(A_{\mathtt{p}}),\mathbb{C}), \quad \textit{if } (|f_{\mathtt{p}}(k^2)|^2)_{k\in\mathbb{N}},\\
f_{\mathtt{a}}(A_{\mathtt{a}}) &\in B(\sigma(A_{\mathtt{a}}),\mathbb{C}), \quad \textit{if } (|f_{\mathtt{a}}([k+\frac{1}{2}]^2)|^2)_{k\in\mathbb{Z}},\\
f_{\mathtt{N}}(A_{\mathtt{N}}) &\in B(\sigma(A_{\mathtt{N}}),\mathbb{C}), \quad \textit{if } (|f_{\mathtt{N}}(k^2)|^2)_{k\in\mathbb{N}},\\
f_{\mathtt{D}}(A_{\mathtt{D}}) &\in B(\sigma(A_{\mathtt{D}}),\mathbb{C}), \quad \textit{if } (|f_{\mathtt{D}}(k^2)|^2)_{k\in\mathbb{N}^*}
\end{aligned}
$$

is summable, then $f_{\mathtt{BC}}(A_{\mathtt{BC}})$ is a Hilbert-Schmidt operator, and hence compact. The latter is the case if

$$|f_{\mathtt{BC}}(\lambda)| \leqslant c\,\lambda^{-\alpha}$$

for every $\lambda\in\sigma(A_{\mathtt{p}}),\ \sigma(A_{\mathtt{a}}),\ \sigma(A_{\mathtt{N}})\setminus\{0\},\ \sigma(A_{\mathtt{D}})$, respectively, where $\alpha>1/2$, $c\geqslant 0$.

3. Canonical Convolutions and Their Integral Representations

The abstract convolution operator given in (1.1) is an infinite series. Integral representation of this series is more convenient for implementation. We provide such representations of $\mathcal{C}*_{\mathtt{BC}}$ corresponding to all types of BC considered. The choice of a Hilbert basis determines an abstract convolution, which we refer to as *canonical*. The most relevant BC in applications are Dirichlet and Neumann BC. In these cases, the connection of the abstract convolution to an integral form is not direct. On the other hand, for periodic and antiperiodic BC, that connection is relatively direct, calling for a periodic and an antiperiodic extension of the micromodulus function, respectively. We define the extension of C to 2-periodic and 2-antiperiodic functions $\widehat{C}_{\mathtt{p}}$ and $\widehat{C}_{\mathtt{a}}$, respectively, by

$$\widehat{C}_{\mathtt{p}}(x+2)=\widehat{C}_{\mathtt{p}}(x),\quad \widehat{C}_{\mathtt{a}}(x+2)=-\widehat{C}_{\mathtt{a}}(x),\quad x\in\mathbb{R}.$$

3.1. **Integral Representation of the Periodic Operator.** We study the properties of the operator $\mathcal{C}*_{\mathtt{p}}$ by starting with its eigenvalues with respect to the associated Hilbert basis $\left(e_k^{\mathtt{p}}\right)_{k\in\mathbb{Z}}$. Considering the eigenfunctions $e_k^{\mathtt{p}}$ and $e_{-k}^{\mathtt{p}}$ for $k\in\mathbb{N}$ of the classical operator $A_{\mathtt{p}}$ and using (1.2), we have

$$\langle e_k^{\mathtt{p}}|C\rangle=\langle e_{-k}^{\mathtt{p}}|C\rangle=\frac{1}{\sqrt{2}}\int_{-1}^{1}\cos(\pi k y)\,C(y)\,dy\ \in\mathbb{R}.$$

Hence, $\mathcal{C}*_{\mathtt{p}}$ is self-adjoint because all members of the sequence $\left(\langle e_k^{\mathtt{p}}|C\rangle\right)_{k\in\mathbb{Z}}$ are real. For $c\in\mathbb{R}$, we conclude that $c-\mathcal{C}*_{\mathtt{p}}$ is a bounded self-adjoint function of $A_{\mathtt{p}}$. Furthermore, if C is in addition positive and

$$c:=\frac{1}{\sqrt{2}}\int_{-1}^{1}C(y)dy,$$

then $c-\mathcal{C}*_{\mathtt{p}}$ becomes positive operator with a spectrum that contains 0.

Next, we present how to obtain an integral representation of $\mathcal{C}*_{\mathtt{p}}$. First note that since

$$(\mathcal{C}*_{\mathtt{p}}u)(x)=\sum_{k\in\mathbb{N}}\langle e_k^{\mathtt{p}}|C\rangle\,\langle e_k^{\mathtt{p}}|u\rangle\ e_k^{\mathtt{p}}(x)=\Big\langle\sum_{k\in\mathbb{N}}(e_k^{\mathtt{p}}(x))^*\ \langle C|e_k^{\mathtt{p}}\rangle\ e_k^{\mathtt{p}}|u\rangle, \tag{3.1}$$

we concentrate on the term $(e_k^{\mathtt{p}}(x))^*\ \langle C|e_k^{\mathtt{p}}\rangle$. We have

$$
\begin{aligned}
(e_k^{\mathtt{p}}(x))^*\ \langle C|e_k^{\mathtt{p}}\rangle &= \frac{1}{2}\,e^{-i\pi kx}\int_{-1}^{1}C^*(y)e^{i\pi ky}\,dy=\frac{1}{2}\int_{-1}^{1}C^*(y)e^{i\pi k(y-x)}\,dy\\
&=\frac{1}{2}\int_{-1+x}^{1+x}\widehat{C}_{\mathtt{p}}^*(y)e^{i\pi k(y-x)}\,dy\ =\frac{1}{2}\int_{-1}^{1}\widehat{C}_{\mathtt{p}}^*(y+x)\,e^{i\pi ky}\,dy\\
&=\frac{1}{2}\int_{-1}^{1}e^{-i\pi ky}\,\widehat{C}_{\mathtt{p}}^*(x-y)\,dy\ =\frac{1}{\sqrt{2}}\,\langle e_k^{\mathtt{p}}|\widehat{C}_{\mathtt{p}}^*(x-\cdot)\rangle\,.
\end{aligned}\tag{3.2}
$$

Since for every finite subset $S\subset\mathbb{N}$,

$$\sum_{k\in S}|(e_k^{\mathtt{p}}(x))^*\ \langle C|e_k^{\mathtt{p}}\rangle\,|^2\leqslant\sum_{k\in S}|\,\langle e_k^{\mathtt{p}}|C\rangle\,|^2\leqslant\sum_{k\in\mathbb{N}}|\,\langle e_k^{\mathtt{p}}|C\rangle\,|^2,$$

the sequence $\left(|(e^{\mathtt{p}}_k(x))^* \ \langle C|e^{\mathtt{p}}_k\rangle\,|^2\right)_{k\in\mathbb{N}}$ is summable. Using (3.2) and for $u \in L^2_{\mathbb{C}}(\Omega)$,

$$\begin{aligned}
\sum_{l\in\mathbb{N}} \langle e^{\mathtt{p}}_{\beta(l)}|C\rangle\,\langle e^{\mathtt{p}}_{\beta(l)}|u\rangle\ e^{\mathtt{p}}_{\beta(l)}(x) &= \Big\langle \sum_{l\in\mathbb{N}} (e^{\mathtt{p}}_{\beta(l)}(x))^*\ \langle C|e^{\mathtt{p}}_{\beta(l)}\rangle\ e^{\mathtt{p}}_{\beta(l)}|u\Big\rangle \\
&= \frac{1}{\sqrt{2}}\Big\langle \sum_{l\in\mathbb{N}} \langle e^{\mathtt{p}}_{\beta(l)}|\widehat{C}^*_{\mathtt{p}}(x-\cdot)\rangle\ e^{\mathtt{p}}_{\beta(l)}|u\Big\rangle \\
&= \frac{1}{\sqrt{2}}\,\langle \widehat{C}^*_{\mathtt{p}}(x-\cdot)|u\rangle\,,
\end{aligned}$$

where $\beta : \mathbb{N} \to \mathbb{Z}$ is some bijection. Consequently, we arrive at the integral representation

$$(\mathcal{C} *_{\mathtt{p}} u)(x) = \frac{1}{\sqrt{2}}\int_{-1}^{1} \widehat{C}_{\mathtt{p}}(x-y)\,u(y)\,dy. \tag{3.3}$$

3.2. **Integral Representation of the Antiperiodic Operator.** We study the properties of the operator $\mathcal{C}*_{\mathtt{a}}$ by starting with its eigenvalues with respect to the associated Hilbert basis $\left(e^{\mathtt{a}}_k\right)_{k\in\mathbb{Z}}$. Considering the eigenfunctions $e^{\mathtt{a}}_k$ and $e^{\mathtt{a}}_{-k}$ for $k \in \mathbb{N}$ of the classical operator $A_{\mathtt{a}}$ and using (1.2), we have

$$\langle e^{\mathtt{a}}_k|C\rangle = \langle e^{\mathtt{a}}_{-k-1}|C\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1}\cos\left[\pi\big(k+\frac{1}{2}\big)y\right]C(y)\,dy\ \in \mathbb{R}.$$

Hence, $\mathcal{C}*_{\mathtt{a}}$ is self-adjoint because all members of the sequence $\left(\langle e^{\mathtt{a}}_k|C\rangle\right)_{k\in\mathbb{Z}}$ are real. For $c \in \mathbb{R}$, we conclude that $c - \mathcal{C}*_{\mathtt{a}}$ is a self-adjoint bounded function of $A_{\mathtt{a}}$. Furthermore, if C is in addition positive and

$$c := \frac{1}{\sqrt{2}}\int_{-1}^{1} C(y)dy,$$

then $c - \mathcal{C}*_{\mathtt{a}}$ becomes a positive operator.

Next, we present how to obtain an integral representation of $\mathcal{C}*_{\mathtt{a}}$. Similar to (3.1), we concentrate on the term $(e^{\mathtt{a}}_k(x))^*\ \langle C|e^{\mathtt{a}}_k\rangle$. We have

$$\begin{aligned}
(e^{\mathtt{a}}_k(x))^*\ \langle C|e^{\mathtt{a}}_k\rangle &= \frac{1}{2}\,e^{-i\pi\left(k+\frac{1}{2}\right)x}\int_{-1}^{1} C^*(y)e^{i\pi\left(k+\frac{1}{2}\right)y}\,dy = \frac{1}{2}\int_{-1}^{1} C^*(y)e^{i\pi\left(k+\frac{1}{2}\right)(y-x)}\,dy \\
&= \frac{1}{2}\int_{-1+x}^{1+x} \widehat{C}^*_{\mathtt{a}}(y)e^{i\pi\left(k+\frac{1}{2}\right)(y-x)}\,dy \qquad = \frac{1}{2}\int_{-1}^{1} \widehat{C}^*_{\mathtt{a}}(y+x)\,e^{i\pi\left(k+\frac{1}{2}\right)y}\,dy \\
&= \frac{1}{2}\int_{-1}^{1} e^{-i\pi\left(k+\frac{1}{2}\right)y}\,\widehat{C}^*_{\mathtt{a}}(x-y)\,dy \qquad = \frac{1}{\sqrt{2}}\,\langle e^{\mathtt{a}}_k|\widehat{C}^*_{\mathtt{a}}(x-\cdot)\rangle\,.
\end{aligned} \tag{3.4}$$

Since for every finite subset $S \subset \mathbb{N}$,

$$\sum_{k\in S} |(e^{\mathtt{a}}_k(x))^*\ \langle C|e^{\mathtt{a}}_k\rangle\,|^2 \leqslant \sum_{k\in S} |\,\langle e^{\mathtt{a}}_k|C\rangle\,|^2 \leqslant \sum_{k\in\mathbb{N}} |\,\langle e^{\mathtt{a}}_k|C\rangle\,|^2,$$

the sequence $\left(|(e^{\mathtt{a}}_k(x))^*\ \langle C|e^{\mathtt{a}}_k\rangle\,|^2\right)_{k\in\mathbb{N}}$ is summable. Using (3.4) and for $u \in L^2_{\mathbb{C}}(\Omega)$,

$$\begin{aligned}
\sum_{l\in\mathbb{N}} \langle e^{\mathtt{a}}_{\beta(l)}|C\rangle\,\langle e^{\mathtt{a}}_{\beta(l)}|u\rangle\ e^{\mathtt{a}}_{\beta(l)}(x) &= \Big\langle \sum_{l\in\mathbb{N}} (e^{\mathtt{a}}_{\beta(l)}(x))^*\ \langle C|e^{\mathtt{a}}_{\beta(l)}\rangle\ e^{\mathtt{a}}_{\beta(l)}|u\Big\rangle \\
&= \frac{1}{\sqrt{2}}\Big\langle \sum_{l\in\mathbb{N}} \langle e^{\mathtt{a}}_{\beta(l)}|\widehat{C}^*_{\mathtt{a}}(x-\cdot)\rangle\ e^{\mathtt{a}}_{\beta(l)}|u\Big\rangle \\
&= \frac{1}{\sqrt{2}}\,\langle \widehat{C}^*_{\mathtt{a}}(x-\cdot)|u\rangle\,,
\end{aligned}$$

where $\beta : \mathbb{N} \to \mathbb{Z}$ is some bijection. Consequently, we arrive at the integral representation

$$(\mathcal{C} *_{\mathtt{a}} u)(x) = \frac{1}{\sqrt{2}}\int_{-1}^{1} \widehat{C}_{\mathtt{a}}(x-y)\,u(y)\,dy. \tag{3.5}$$

3.3. **Integral Representation of the Neumann Operator.** We study the properties of the operator $\mathcal{C}*_{\mathtt{N}}$ by starting with its eigenvalues with respect to the associated Hilbert basis $\big(e_k^{\mathtt{N}}\big)_{k\in\mathbb{N}^*}$. Considering the eigenfunctions $e_k^{\mathtt{N}}$ for $k \in \mathbb{N}$ of the classical operator $A_{\mathtt{N}}$ and using (1.2), we have

$$\langle e_k^{\mathtt{N}}|C\rangle = \int_{-1}^{1} \cos\big(\frac{k\pi}{2}(y+1)\big)C(y)\,dy \;\in \mathbb{R}$$

Hence, $\mathcal{C}*_{\mathtt{N}}$ is self-adjoint because all members of the sequence $\big(\langle e_k^{\mathtt{N}}|C\rangle\big)_{k\in\mathbb{N}}$ are real. For $c \in \mathbb{R}$, we conclude that $c - \mathcal{C}*_{\mathtt{N}}$ is a self-adjoint bounded function of $A_{\mathtt{N}}$. Furthermore, if C is in addition positive and

$$c := \int_{-1}^{1} C(y)dy,$$

by observing

$$\int_{-1}^{1} C(y)\,dy - \langle e_k^{\mathtt{N}}|C\rangle = \int_{-1}^{1} \big[1-\cos\big(\frac{k\pi}{2}(y+1)\big)\big]C(y)\,dy, \quad k \in \mathbb{N}^*,$$

we conclude that $c - \mathcal{C}*_{\mathtt{N}}$ is a positive operator.

Next, we present how to obtain an integral representation of $\mathcal{C}*_{\mathtt{N}}$. Similar to (3.1), we concentrate on the term $(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k^{\mathtt{N}}\rangle$. We have $\langle e_k^{\mathtt{N}}|C\rangle = 0$, for every odd $k \in \mathbb{N}$ because C is even. Since for every finite subset $S \subset \mathbb{N}$

$$\sum_{k\in S} |(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k^{\mathtt{N}}\rangle\,|^2 \leqslant \sum_{k\in S} |\,\langle e_k^{\mathtt{N}}|C\rangle\,|^2 \leqslant \sum_{k\in\mathbb{N}} |\,\langle e_k^{\mathtt{N}}|C\rangle\,|^2,$$

the sequence $\big(|(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k^{\mathtt{N}}\rangle\,|^2\big)_{k\in\mathbb{N}}$ is summable.

Since for $k \in \mathbb{N}$, $x, y \in \mathbb{R}$

$$\begin{aligned}\cos\big(\frac{k\pi}{2}(x+1)\big)\cos\big(\frac{k\pi}{2}(y+1)\big) \;&=\; \frac{1}{2}\big\{\big[\cos\big(\frac{k\pi}{2}(x-y+1)\big)+\cos\big(\frac{k\pi}{2}(x+y+1)\big)\big]\cos\big(\frac{k\pi}{2}\big)\\ &\qquad +\big[\sin\big(\frac{k\pi}{2}(x-y+1)\big)-\sin\big(\frac{k\pi}{2}(x+y+1)\big)\big]\sin\big(\frac{k\pi}{2}\big)\big\}\,,\end{aligned}$$

the expression of $(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k\rangle$ reduces to an expression that involves only cosine terms. More precisely, for even $k \in \mathbb{N}^*$, we have

$$\begin{aligned}(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k^{\mathtt{N}}\rangle \;&=\; \cos\big(\frac{k\pi}{2}(x+1)\big)\int_{-1}^{1} C^*(y)\cos\big(\frac{k\pi}{2}(y+1)\big)dy\\ &=\; \frac{1}{2}\cos\big(\frac{k\pi}{2}\big)\big[\int_{-1}^{1} C^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy+\int_{-1}^{1} C^*(y)\,\cos\big(\frac{k\pi}{2}(x+y+1)\big)dy\big]\\ &=\; \cos\big(\frac{k\pi}{2}\big)\int_{-1}^{1} C^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy \qquad (3.6)\\ &=\; \cos\big(\frac{k\pi}{2}\big)\int_{x-1}^{x+1} \widehat{C}^*_{\mathtt{p}}(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy \qquad (3.7)\\ &=\; \cos\big(\frac{k\pi}{2}\big)\int_{-1}^{1} \widehat{C}^*_{\mathtt{p}}(x-y)\,\cos\big(\frac{k\pi}{2}(y+1)\big)dy\\ &=\; \cos\big(\frac{k\pi}{2}\big)\,\langle e_k^{\mathtt{N}}|\widehat{C}^*_{\mathtt{p}}(x-\mathrm{id}_{\mathbb{R}})\rangle\,. \qquad (3.8)\end{aligned}$$

Here, it has been used that $\cos\big(\frac{k\pi}{2}(\mathrm{id}_{\mathbb{R}}+1)\big)$ is 2-*periodic* for even $k \in \mathbb{N}^*$.

Remark 3.1. *For* $\mathtt{BC} = \mathtt{p}, \mathtt{a}$*, the critical step in obtaining an integral representation of* $(\mathcal{C} *_{\mathtt{BC}} u)(x)$ *is connecting the term* $(e_k^{\mathtt{BC}}(x))^*\ \langle C|e_k^{\mathtt{BC}}\rangle$ *to the inner product* $\langle e_k^{\mathtt{BC}}|\widehat{C}^*_{\mathtt{BC}}(x-\cdot)\rangle$*; see the last steps of periodic and antiperiodic derivation in* (3.2) *and* (3.4)*, respectively. More concisely, integral representation is obtained by following the steps below.*

$$(\mathcal{C} *_{\mathtt{BC}} u)(x) \;=\; \sum_{k\in\mathbb{N}} \langle e_k^{\mathtt{BC}}|C\rangle\,\langle e_k^{\mathtt{BC}}|u\rangle\; e_k^{\mathtt{BC}}(x)$$

$$
\begin{aligned}
&= \Big\langle \sum_{k\in\mathbb{N}} (e_k^{\mathtt{BC}}(x))^* \, \langle C|e_k^{\mathtt{BC}}\rangle \; e_k^{\mathtt{BC}} \Big| u \Big\rangle \\
&= \frac{1}{\sqrt{2}} \Big\langle \sum_{k\in\mathbb{N}} \langle e_k^{\mathtt{BC}}|\widehat{C}^*_{\mathtt{BC}}(x-\cdot)\rangle \; e_k^{\mathtt{BC}} \Big| u \Big\rangle \\
&= \frac{1}{\sqrt{2}} \langle \widehat{C}^*_{\mathtt{BC}}(x-\cdot)|u\rangle
\end{aligned}
$$

In step (3.8), the term $\cos\left(\frac{k\pi}{2}\right)$ *prevents this connection. We need to connect to the inner product of another function. Next, we pursue what that function should be. This is a nontrivial task and by the help of "half-wave symmetry", we identify one such function.*

We note for $k \in \mathbb{N}^*$ that

$$
\cos\big(\frac{k\pi}{2}\big) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even and } k/2 \text{ is even,} \\ -1 & \text{if } k \text{ is even and } k/2 \text{ is odd.} \end{cases}
$$

Next, we decompose C into $C_1, C_2 \in L^2(\Omega)$ defined by

$$
C_1(x) := \frac{1}{2}\big[C(|x|) + C(1-|x|)\big], \; C_2(x) := \frac{1}{2}\big[C(|x|) - C(1-|x|)\big]. \tag{3.9}
$$

Then, it is easy to check that

$$
C = C_1 + C_2, \tag{3.10}
$$

and that they are even functions which satisfy a "half-wave symmetry" property, i.e.,

$$
C_1(1-x) = C_1(x) \text{ and } C_2(1-x) = -C_2(x), \quad x \in [0,1/2].
$$

As a consequence, for even $k \in \mathbb{N}^*$ and $j = 1, 2$,

$$
\begin{aligned}
\langle e_k^{\mathtt{N}}|C_j\rangle &= \int_{-1}^{1} \cos\big(\frac{k\pi}{2}(y+1)\big) C_j(y)\, dy \\
&= \int_{-1}^{1} \big[\cos\big(\frac{k\pi}{2}y\big)\cos\big(\frac{k\pi}{2}\big) - \sin\big(\frac{k\pi}{2}y\big)\sin\big(\frac{k\pi}{2}\big)\big] C_j(y)\, dy \\
&= \cos\big(\frac{k\pi}{2}\big) \int_{-1}^{1} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy = 2\cos\big(\frac{k\pi}{2}\big) \int_{0}^{1} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy \\
&= 2\cos\big(\frac{k\pi}{2}\big)\Big[\int_0^{1/2} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy + \int_{1/2}^{1} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy\Big] \\
&= 2\cos\big(\frac{k\pi}{2}\big)\Big[\int_0^{1/2} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy + \int_{0}^{1/2} \cos\big(\frac{k\pi}{2}(1-y)\big) C_j(1-y)\, dy\Big] \\
&= 2\cos\big(\frac{k\pi}{2}\big)\Big[\int_0^{1/2} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy + (-1)^{j+1}(-1)^{k/2} \int_{0}^{1/2} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy\Big] \\
&= 2\cos\big(\frac{k\pi}{2}\big)\big[1 + (-1)^{j+1}(-1)^{k/2}\big] \int_0^{1/2} \cos\big(\frac{k\pi}{2}y\big) C_j(y)\, dy,
\end{aligned}
$$

and hence

$$
\langle e_k^{\mathtt{N}}|C_j\rangle = 0 \text{ if } \begin{cases} k \text{ is odd,} \\ k \text{ is even, } k/2 \text{ is odd and } j = 1, \\ k \text{ is even, } k/2 \text{ is even and } j = 2. \end{cases} \tag{3.11}
$$

For even $k \in \mathbb{N}^*$, using (3.10), we simply have

$$
(e_k^{\mathtt{N}}(x))^* \, \langle C|e_k^{\mathtt{N}}\rangle = (e_k^{\mathtt{N}}(x))^* \, \langle C_1|e_k^{\mathtt{N}}\rangle + (e_k^{\mathtt{N}}(x))^* \, \langle C_2|e_k^{\mathtt{N}}\rangle\,. \tag{3.12}
$$

The equation (3.6) holds for any even kernel function. We write (3.6) for $C_j,\ j=1,2$, which yields

$$(e_k^{\mathtt{N}}(x))^*\ \langle C_j|e_k^{\mathtt{N}}\rangle = \cos\big(\frac{k\pi}{2}\big)\int_{-1}^{1} C_j^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy. \tag{3.13}$$

Using (3.13) and (3.11), we obtain the following equations, if

$$k/2 \text{ is even},\quad (e_k^{\mathtt{N}}(x))^*\ \langle C_2|e_k^{\mathtt{N}}\rangle = 0,\quad (e_k^{\mathtt{N}}(x))^*\ \langle C_1|e_k^{\mathtt{N}}\rangle = \int_{-1}^{1} C_1^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy, \tag{3.14}$$

$$k/2 \text{ is odd},\quad (e_k^{\mathtt{N}}(x))^*\ \langle C_1|e_k^{\mathtt{N}}\rangle = 0,\quad (e_k^{\mathtt{N}}(x))^*\ \langle C_2|e_k^{\mathtt{N}}\rangle = -\int_{-1}^{1} C_2^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy. \tag{3.15}$$

Combining (3.14) and (3.15), for any even $k\in\mathbb{N}^*$, we arrive at

$$\begin{aligned}(e_k^{\mathtt{N}}(x))^*\ \langle C_1|e_k^{\mathtt{N}}\rangle + (e_k^{\mathtt{N}}(x))^*\ \langle C_2|e_k^{\mathtt{N}}\rangle &= \\ \int_{-1}^{1} C_1^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy &- \int_{-1}^{1} C_2^*(y)\,\cos\big(\frac{k\pi}{2}(x-y+1)\big)dy.\end{aligned} \tag{3.16}$$

Now, applying the extension argument in (3.7) for the terms in (3.16), we obtain

$$(e_k^{\mathtt{N}}(x))^*\ \langle C_1|e_k^{\mathtt{N}}\rangle + (e_k^{\mathtt{N}}(x))^*\ \langle C_2|e_k^{\mathtt{N}}\rangle = \langle e_k^{\mathtt{N}}|\widehat{C}_{1,\mathtt{p}}^*(x-\mathrm{id}_{\mathbb{R}})\rangle - \langle e_k^{\mathtt{N}}|\widehat{C}_{2,\mathtt{p}}^*(x-\mathrm{id}_{\mathbb{R}})\rangle\,, \tag{3.17}$$

where $\widehat{C}_{j,\mathtt{p}}$ denotes the extension of C_j to a 2-*periodic* function on $\mathbb{R}$. Consequently, (3.17) and (3.12) yield

$$(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k^{\mathtt{N}}\rangle = \langle e_k^{\mathtt{N}}|\widehat{C}_{1,\mathtt{p}}^*(x-\mathrm{id}_{\mathbb{R}})\rangle - \langle e_k^{\mathtt{N}}|\widehat{C}_{2,\mathtt{p}}^*(x-\mathrm{id}_{\mathbb{R}})\rangle\,. \tag{3.18}$$

In the following, we extend (3.12) to odd $k\in\mathbb{N}^*$. For this purpose, we note for even $u\in L^2_{\mathbb{C}}(\Omega)$ that its 2-periodic extension $\hat{u}_{\mathtt{p}}$ is also even since for $l\in\mathbb{N}$ and $x\in[-1-2l,1-2l]$, we have

$$\hat{u}_{\mathtt{p}}(x) = u(x+2l) = u(-x-2l) = \hat{u}_{\mathtt{p}}(-x-2l+2l) = \hat{u}_{\mathtt{p}}(-x).$$

Hence, it follows for even u and odd $k\in\mathbb{N}^*$ that

$$\langle e_k^{\mathtt{N}}|\hat{u}_{\mathtt{p}}(x-\cdot)\rangle = -\langle e_k^{\mathtt{N}}|\hat{u}_{\mathtt{p}}(-x-\cdot)\rangle\,,$$

which implies that

$$\langle e_k^{\mathtt{N}}|\hat{u}_{\mathtt{p}}(x-\cdot)+\hat{u}_{\mathtt{p}}(-x-\cdot)\rangle = 0.$$

On the other hand, for even $k\in\mathbb{N}^*$

$$\langle e_k^{\mathtt{N}}|\hat{u}_{\mathtt{p}}(x-\cdot)\rangle = \langle e_k^{\mathtt{N}}|\hat{u}_{\mathtt{p}}(-x-\cdot)\rangle\,.$$

Consequently,

$$\langle e_k^{\mathtt{N}}|\frac{1}{2}\left[\hat{u}_{\mathtt{p}}(x-\cdot)+\hat{u}_{\mathtt{p}}(-x-\cdot)\right]\rangle = \begin{cases} 0 & \text{if } k\in\mathbb{N}^* \text{ is odd},\\ \langle e_k^{\mathtt{N}}|\hat{u}_{\mathtt{p}}(x-\cdot)\rangle & \text{if } k\in\mathbb{N}^* \text{ is even}.\end{cases}$$

Therefore, for $k\in\mathbb{N}^*$, we conclude from (3.12) and (3.17) that

$$(e_k^{\mathtt{N}}(x))^*\ \langle C|e_k^{\mathtt{N}}\rangle = \langle e_k^{\mathtt{N}}|\frac{1}{2}\left[\widehat{C}_{1,\mathtt{p}}^*(x-\cdot)+\widehat{C}_{1,\mathtt{p}}(-x-\cdot)-\widehat{C}_{2,\mathtt{p}}^*(x-\cdot)-\widehat{C}_{2,\mathtt{p}}^*(-x-\cdot)\right]\rangle\,.$$

For $k=0$, we have

$$\begin{aligned}&\langle e_0^{\mathtt{N}}|\frac{1}{2}\left[\widehat{C}_{1,\mathtt{p}}^*(x-\cdot)+\widehat{C}_{1,\mathtt{p}}(-x-\cdot)-\widehat{C}_{2,\mathtt{p}}^*(x-\cdot)-\widehat{C}_{2,\mathtt{p}}^*(-x-\cdot)\right]\rangle\\ &= 2^{-3/2}\Big[\int_{-1}^{1}\widehat{C}_{1,\mathtt{p}}^*(x-y)\,dy+\int_{-1}^{1}\widehat{C}_{1,\mathtt{p}}^*(-x-y)\,dy-\int_{-1}^{1}\widehat{C}_{2,\mathtt{p}}^*(x-y)\,dy-\int_{-1}^{1}\widehat{C}_{2,\mathtt{p}}^*(-x-y)]\,dy\Big]\\ &= 2^{-3/2}\Big[\int_{-1}^{1}\widehat{C}_{1,\mathtt{p}}^*(y-x)\,dy+\int_{-1}^{1}\widehat{C}_{1,\mathtt{p}}^*(y+x)\,dy-\int_{-1}^{1}\widehat{C}_{2,\mathtt{p}}^*(y-x)\,dy-\int_{-1}^{1}\widehat{C}_{2,\mathtt{p}}^*(y+x)\,dy\Big]\\ &= 2^{-3/2}\Big[\int_{-1}^{1}C_1^*(y)\,dy+\int_{-1}^{1}C_1^*(y)\,dy-\int_{-1}^{1}C_2^*(y)\,dy-\int_{-1}^{1}C_2^*(y)\,dy\Big]\\ &= 2^{-1/2}\Big[\int_{-1}^{1}C_1^*(y)\,dy-\int_{-1}^{1}C_2^*(y)\,dy\Big]\end{aligned}$$

$$= (e_0^{\mathtt{N}}(x))^* \ \langle C|e_0^{\mathtt{N}}\rangle + \frac{\sqrt{2}-1}{2} \int_{-1}^{1} C_1^*(y)\, dy - \frac{\sqrt{2}+1}{2} \int_{-1}^{1} C_2^*(y)\, dy.$$

Hence

$$(e_0^{\mathtt{N}}(x))^* \ \langle C|e_0^{\mathtt{N}}\rangle \, e_0^{\mathtt{N}} = \langle e_0^{\mathtt{N}}|\frac{1}{2}\, [\widehat{C}_{1,\mathtt{p}}^*(x-\cdot) + \widehat{C}_{1,\mathtt{p}}^*(-x-\cdot) - \widehat{C}_{2,\mathtt{p}}^*(x-\cdot) - \widehat{C}_{2,\mathtt{p}}^*(-x-\cdot)]\rangle \, e_0^{\mathtt{N}} + \gamma_{\mathtt{N},C} e_0^{\mathtt{N}},$$

where

$$\gamma_{\mathtt{N},C} := -\frac{\sqrt{2}-1}{2} \int_{-1}^{1} C_1^*(y)\, dy + \frac{\sqrt{2}+1}{2} \int_{-1}^{1} C_2^*(y)\, dy.$$

Hence, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{N}} \langle e_k^{\mathtt{N}}|C\rangle \, \langle e_k^{\mathtt{N}}|u\rangle \ e_k^{\mathtt{N}}(x) &= \big\langle \sum_{k\in\mathbb{N}} (e_k^{\mathtt{N}}(x))^* \ \langle C|e_k^{\mathtt{N}}\rangle \ e_k^{\mathtt{N}}|u\big\rangle \\ &= \big\langle \sum_{k\in\mathbb{N}} \langle e_k^{\mathtt{N}}|\frac{1}{2}\, [\widehat{C}_{1,\mathtt{p}}^*(x-\cdot) + \widehat{C}_{1,\mathtt{p}}^*(-x-\cdot) - \widehat{C}_{2,\mathtt{p}}^*(x-\cdot) - \widehat{C}_{2,\mathtt{p}}^*(-x-\cdot)]\rangle \, e_k^{\mathtt{N}}|u\big\rangle + \gamma_{N,C}\, \langle e_0^{\mathtt{N}}|u\rangle\end{aligned}$$

Consequently, we arrive at the integral representation

$$(\mathcal{C} *_{\mathtt{N}} u)(x) = \frac{1}{2}\int_{-1}^{1} \big[\widehat{C}_{1,\mathtt{p}}(x-y) + \widehat{C}_{1,\mathtt{p}}(-x-y) - \widehat{C}_{2,\mathtt{p}}(x-y) - \widehat{C}_{2,\mathtt{p}}(-x-y)\big]\, u(y)\, dy + \gamma_{N,C}\, \langle e_0^{\mathtt{N}}|u\rangle\, .$$

Recalling the definitions of C_1 and C_2 in (3.9) and employing a change of variable, we obtain a more compact integral representation

$$(\mathcal{C} *_{\mathtt{N}} u)(x) = \int_{-1}^{1} \widehat{C}_{\mathtt{p}}(|x-y|-1)\, P_e u(y)\, dy + \gamma_{N,C}\, \langle e_0^{\mathtt{N}}|u\rangle\, ,$$

where P_e denotes the even part of a function whose definition is given in (4.1).

3.4. **Integral Representation of the Dirichlet Operator.** We study the properties of the operator $\mathcal{C}*_{\mathtt{D}}$ by starting with its eigenvalues with respect to the associated Hilbert basis $\big(e_k^{\mathtt{D}}\big)_{k\in\mathbb{N}^*}$. Considering the eigenfunctions $e_k^{\mathtt{D}}$ for $k\in\mathbb{N}^*$ of the classical operator $A_{\mathtt{D}}$ and using (1.2), we have

$$\langle e_k^{\mathtt{D}}|C\rangle = \int_{-1}^{1} \sin\big(\frac{k\pi}{2}(y+1)\big) C(y)\, dy.$$

Hence, $\mathcal{C}*_{\mathtt{D}}$ is self-adjoint because all members of the sequence $\big(\, \langle e_k^{\mathtt{D}}|C\rangle\, \big)_{k\in\mathbb{N}^*}$ are real. For $c\in\mathbb{R}$, we conclude that $c-\mathcal{C}*_{\mathtt{D}}$ is a self-adjoint bounded function of $A_{\mathtt{D}}$. Furthermore, if C is in addition positive and

$$c := \int_{-1}^{1} C(y) dy,$$

by observing

$$\int_{-1}^{1} C\, dy - \langle e_k^{\mathtt{D}}|C\rangle = \int_{-1}^{1} \big[1-\sin\big(\frac{k\pi}{2}(y+1)\big)\big] C(y)\, dy, \quad k\in\mathbb{N}^*,$$

we conclude that $c-\mathcal{C}*_{\mathtt{D}}$ is a positive operator.

Next, we present how to obtain an integral representation of $\mathcal{C}*_{\mathtt{D}}$. Similar to (3.1), we concentrate on the term $(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle$. We have $\langle e_k^{\mathtt{D}}|C\rangle = 0$ for every even $k\in\mathbb{N}^*$ because C is even. Since for every finite subset $S\subset\mathbb{N}^*$

$$\sum_{k\in S} |(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle\, |^2 \leqslant \sum_{k\in S} |\, \langle e_k^{\mathtt{D}}|C\rangle\, |^2 \leqslant \sum_{k\in\mathbb{N}^*} |\, \langle e_k^{\mathtt{D}}|C\rangle\, |^2,$$

the sequence $\big(|(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle\, |^2\big)_{k\in\mathbb{N}^*}$ is summable.

Since for $k\in\mathbb{N}^*$, $x,y\in\mathbb{R}$

$$\begin{aligned}\sin\big(\frac{k\pi}{2}(x+1)\big)\sin\big(\frac{k\pi}{2}(y+1)\big) \quad &= \quad \frac{1}{2}\big\{\big[\cos\big(\frac{k\pi}{2}(x-y+1)\big) - \cos\big(\frac{k\pi}{2}(x+y+1)\big)\big]\cos\big(\frac{k\pi}{2}\big)\big] \\ &\quad +\big[\sin\big(\frac{k\pi}{2}(x-y+1)\big) + \sin\big(\frac{k\pi}{2}(x+y+1)\big)\big]\sin\big(\frac{k\pi}{2}\big)\big\},\end{aligned}$$

the expression of $(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle$ reduces to an expression that involves only sine terms. More precisely, for odd $k\in\mathbb{N}^*$, we have

$$\begin{aligned}(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle &= \sin\big(\frac{k\pi}{2}(x+1)\big)\int_{-1}^{1} C^*(y)\sin\big(\frac{k\pi}{2}(y+1)\big)dy\\ &= \sin\big(\frac{k\pi}{2}\big)\int_{-1}^{1} C^*(y)\ \sin\big(\frac{k\pi}{2}(x-y+1)\big)dy\\ &= \sin\big(\frac{k\pi}{2}\big)\int_{x-1}^{x+1} \widehat{C}_{\mathtt{a}}^*(y)\ \sin\big(\frac{k\pi}{2}(x-y+1)\big)dy\\ &= \sin\big(\frac{k\pi}{2}\big)\int_{-1}^{1} \widehat{C}_{\mathtt{a}}^*(x-y)\ \sin\big(\frac{k\pi}{2}(y+1)\big)dy\\ &= \sin\big(\frac{k\pi}{2}\big)\,\langle e_k^{\mathtt{D}}|\widehat{C}_{\mathtt{a}}^*(x-\mathrm{id}_{\mathbb{R}})\rangle\,. \end{aligned} \tag{3.19}$$

Here, it has been used that $\sin\big(\frac{k\pi}{2}(\mathrm{id}_{\mathbb{R}}+1)\big)$ is 2-*antiperiodic* for odd $k\in\mathbb{N}^*$.

Remark 3.2. *Similar to the Neumann case discussed in Remark 3.1, the term* $\sin\big(\frac{k\pi}{2}\big)$ *in* (3.19) *prevents us from connecting the term* $(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle$ *to the inner product* $\langle e_k^{\mathtt{D}}|\widehat{C}_{\mathtt{D}}^*(x-\cdot)\rangle$*. Thus, we pursue an alternative. It turns out that this alternative involves a projection which has a limit expression and it is presented in Sec. 3.4.1.*

We note for $k\in\mathbb{N}^*$ that

$$\sin\big(\frac{k\pi}{2}\big)=\begin{cases}0 & \text{if } k \text{ is even,}\\ 1 & \text{if } k \text{ is odd and } (k-1)/2 \text{ is even,}\\ -1 & \text{if } k \text{ is odd and } (k-1)/2 \text{ is odd.}\end{cases}$$

As a consequence, if we denote by P the orthogonal projection onto the closure of the subspace

$$\mathrm{Span}(\{e_{4l+1}^{\mathtt{D}} : l\in\mathbb{N}\}),$$

then for odd $k\in\mathbb{N}^*$

$$\begin{aligned}(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle &= (e_k^{\mathtt{D}}(x))^* \,\langle \mathrm{P}C|e_k^{\mathtt{D}}\rangle + (e_k^{\mathtt{D}}(x))^* \,\langle C-\mathrm{P}C|e_k^{\mathtt{D}}\rangle\\ &= \langle e_k^{\mathtt{D}}|(\widehat{\mathrm{P}C}_{\mathtt{a}})^*(x-\cdot) - (\widehat{C}_{\mathtt{a}}-\widehat{\mathrm{P}C}_{\mathtt{a}})^*(x-\cdot)\rangle\,,\end{aligned} \tag{3.20}$$

where $\widehat{\mathrm{P}C}_{\mathtt{a}}$ denotes the extension of PC to a 2-*antiperiodic* function on $\mathbb{R}$. Here, we used the fact that $e_k^{\mathtt{D}}$ is even and hence so are PC and $C-\mathrm{P}C$.

In the following, we extend (3.20) to even $k\in\mathbb{N}^*$. For this purpose, we note for every even $u\in L^2_{\mathbb{C}}(\Omega)$ that its 2-anti-periodic extension $\hat{u}_{\mathtt{a}}$ is also even since for $l\in\mathbb{N}$ and $x\in[-1-2l,-2l]$, we have

$$\hat{u}_{\mathtt{a}}(x)=(-1)^l u(x+2l)=(-1)^l u(-x-2l)=(-1)^l\,(-1)^l\hat{u}_{\mathtt{a}}(-x-2l+2l)=\hat{u}_{\mathtt{a}}(-x).$$

Hence it follows for even u and even $k\in\mathbb{N}^*$ that

$$\langle e_k^{\mathtt{D}}|\hat{u}_{\mathtt{a}}(x-\cdot)\rangle = -\,\langle e_k^{\mathtt{D}}|\hat{u}_{\mathtt{a}}(-x-\cdot)\rangle\,,$$

which implies that

$$\langle e_k^{\mathtt{D}}|\hat{u}_{\mathtt{a}}(x-\cdot)+\hat{u}_{\mathtt{a}}(-x-\cdot)\rangle = 0.$$

On the other hand, for odd $k\in\mathbb{N}^*$

$$\langle e_k^{\mathtt{D}}|\hat{u}_{\mathtt{a}}(x-\cdot)\rangle = \langle e_k^{\mathtt{D}}|\hat{u}_{\mathtt{a}}(-x-\cdot)\rangle\,.$$

Consequently,

$$\langle e_k^{\mathtt{D}}|\frac{1}{2}\,[\hat{u}_{\mathtt{a}}(x-\cdot)+\hat{u}_{\mathtt{a}}(-x-\cdot)]\rangle = \begin{cases}0 & \text{if } k\in\mathbb{N}^* \text{ is even,}\\ \langle e_k^{\mathtt{D}}|\hat{u}_{\mathtt{a}}(x-\cdot)\rangle & \text{if } k\in\mathbb{N}^* \text{ is odd.}\end{cases}$$

Therefore, we conclude from (3.20) for $k\in\mathbb{N}^*$ that

$$(e_k^{\mathtt{D}}(x))^* \ \langle C|e_k^{\mathtt{D}}\rangle$$

$$= \langle e^{\mathtt{D}}_k | \frac{1}{2} \left[(\widehat{\mathrm{PC}}_{\mathtt{a}})^*(x - \cdot) + (\widehat{\mathrm{PC}}_{\mathtt{a}})^*(-x - \cdot) - (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})^*(x - \cdot) - (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})^*(-x - \cdot) \right] \rangle .$$

Hence, we obtain

$$\begin{aligned}
\sum_{k \in \mathbb{N}^*} \langle e^{\mathtt{D}}_k | C \rangle \, \langle e^{\mathtt{D}}_k | u \rangle \; e^{\mathtt{D}}_k(x) &= \Big\langle \sum_{k \in \mathbb{N}^*} (e^{\mathtt{D}}_k(x))^* \; \langle C | e^{\mathtt{D}}_k \rangle \; e^{\mathtt{D}}_k | u \Big\rangle \\
&= \Big\langle \sum_{k \in \mathbb{N}^*} \langle e^{\mathtt{D}}_k | \frac{1}{2} \left[(\widehat{\mathrm{PC}}_{\mathtt{a}})^*(x - \cdot) + (\widehat{\mathrm{PC}}_{\mathtt{a}})^*(-x - \cdot) - (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})^*(x - \cdot) - (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})^*(-x - \cdot) \right] \rangle \, e^{\mathtt{D}}_k | u \Big\rangle \\
&= \frac{1}{2} \left[\langle (\widehat{\mathrm{PC}}_{\mathtt{a}})^*(x - \cdot) | u \rangle + \langle (\widehat{\mathrm{PC}}_{\mathtt{a}})^*(-x - \cdot) | u \rangle - \langle (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})^*(x - \cdot) | u \rangle - \langle (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})^*(-x - \cdot) | u \rangle \right].
\end{aligned}$$

Consequently, we arrive at the integral representation

$$(\mathcal{C} *_{\mathtt{D}} u)(x) = \frac{1}{2} \int_{-1}^{1} \left[(\widehat{\mathrm{PC}}_{\mathtt{a}})(x - y) + (\widehat{\mathrm{PC}}_{\mathtt{a}})(-x - y) - (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})(x - y) - (\widehat{C}_{\mathtt{a}} - \widehat{\mathrm{PC}}_{\mathtt{a}})(-x - y) \right] u(y) \, dy.$$

3.4.1. *Representation of the Projection Present in the Canonical Convolution.* We give a representation of P that is independent of the orthonormal basis used in its definition. We note for $x \in \Omega$ that

$$\sum_{k=0}^{n} \langle e^{\mathtt{D}}_{4k+1} | u \rangle \, e^{\mathtt{D}}_{4k+1}(x) = \Big\langle \sum_{k=0}^{n} e^{\mathtt{D}}_{4k+1}(x) \, e^{\mathtt{D}}_{4k+1} | u \Big\rangle,$$

and for $y \in \Omega$ that

$$\begin{aligned}
e^{\mathtt{D}}_{4k+1}(x) \, e^{\mathtt{D}}_{4k+1}(y) &= \sin\Big(\frac{(4k+1)\pi}{2} (x+1) \Big) \sin\Big(\frac{(4k+1)\pi}{2} (y+1) \Big) \\
&= \frac{1}{2} \Big[\sin\Big(\frac{(4k+1)\pi}{2} (x - y + 1) \Big) + \sin\Big(\frac{(4k+1)\pi}{2} (x + y + 1) \Big) \Big].
\end{aligned}$$

For $a \in \mathbb{R}$, we have

$$\begin{aligned}
\sum_{k=0}^{n} \sin((4k+1)a) &= \sum_{k=0}^{n} [\sin(4ka)\cos(a) + \cos(4ka)\sin(a)] \\
&= \cos(a) \sum_{k=0}^{n} \sin(4ka) + \sin(a) \sum_{k=0}^{n} \cos(4ka). \qquad (3.21)
\end{aligned}$$

For $n \in \mathbb{N}^*$, $b \in \mathbb{C}$ satisfying $b \neq 2\pi l, l \in \mathbb{Z}$, we also have

$$\begin{aligned}
\sum_{k=0}^{n} \sin(kb) &= \frac{1}{2i} \Big[\sum_{k=0}^{n} e^{ikb} - \sum_{k=0}^{n} e^{-ikb} \Big] = \frac{1}{2i} \Big[\sum_{k=0}^{n} (e^{ib})^k - \sum_{k=0}^{n} (e^{-ib})^k \Big] \\
&= \frac{-\sin(b(n+1)) + \sin(bn) + \sin(b)}{4\sin^2(\frac{b}{2})} \qquad (3.22) \\
\sum_{k=0}^{n} \cos(kb) &= \frac{1}{2} \Big[\sum_{k=0}^{n} e^{ikb} + \sum_{k=0}^{n} e^{-ikb} \Big] = \frac{1}{2} \Big[\sum_{k=0}^{n} (e^{ib})^k + \sum_{k=0}^{n} (e^{-ib})^k \Big] \\
&= \frac{-\cos(b(n+1)) + \cos(bn) - \cos(b) + 1}{4\sin^2(\frac{b}{2})}. \qquad (3.23)
\end{aligned}$$

We substitute $b = 4a$ in (3.22) and (3.23) and plug these expressions in (3.21). For $a \neq l\pi/2, l \in \mathbb{Z}$, (3.21) turns into the following formula

$$\sum_{k=0}^{n} \sin((4k+1)a) = \frac{-\sin(a(4n+5)) + \sin(a(4n+1)) + \sin(3a) + \sin(a)}{4\sin^2(2a)}.$$

For $x - y, \; x + y \notin \mathbb{Z}$, let

$$K_n(x-y) := \frac{1}{4\sin^2(\pi(x-y+1))} \Big[-\sin\Big(\frac{\pi(4n+5)}{2} (x-y+1) \Big) + \sin\Big(\frac{\pi(4n+1)}{2} (x-y+1) \Big)$$

$$+ \sin\big(\frac{3\pi}{2}(x-y+1)\big) + \sin\big(\frac{\pi}{2}(x-y+1)\big)\big]. \tag{3.24}$$

Then, the expression of (3.21) can be simplified as

$$\sum_{k=0}^{n} e^{\mathtt{D}}_{4k+1}(x)\, e^{\mathtt{D}}_{4k+1}(y) = \frac{1}{2}\, K_n(x-y) + \frac{1}{2}\, K_n(x+y).$$

Consequently,

$$\sum_{k=0}^{n} \langle e^{\mathtt{D}}_{4k+1}|u\rangle\, e^{\mathtt{D}}_{4k+1}(x) = \frac{1}{2}\int_{-1}^{1} K_n(x-y)\, u(y)\, dy + \frac{1}{2}\int_{-1}^{1} K_n(x+y)\, u(y)\, dy.$$

After a change of variable in the second integral, we obtain

$$\sum_{k=0}^{n} \langle e^{\mathtt{D}}_{4k+1}|u\rangle\, e^{\mathtt{D}}_{4k+1}(x) = \int_{-1}^{1} K_n(x-y)\, \frac{1}{2}\,[u(y)+u(-y)]\, dy.$$

Hence, we arrive at the integral representation of P

$$\mathrm{P}u = \lim_{n\to\infty} K_n * \frac{1}{2}\,[u + u\circ(-\mathrm{id}_\Omega)],$$

where K_n is given in (3.24), $*$ denotes the integral convolution on Ω, and the limit is to be performed in $L^2_{\mathbb{C}}(\Omega)$.

4. Alternative Governing Operators

The main property we exploit in satisfying the BC is the evenness of the kernel function. Inspired by this fact, we can define alternative governing operators that are structurally simpler than $\mathcal{C}*_{\mathtt{N}}$ and $\mathcal{C}*_{\mathtt{D}}$ that satisfy homogeneous Neumann and Dirichlet BC, respectively. We will call these operators *simple* convolutions. Employing even kernel functions, these are derived from certain combinations of convolutions that satisfy periodic and antiperiodic BC together with even and odd input functions. Since even and odd parts of an input function is used in the definition of simple convolution operators, here we provide their definitions. We denote the orthogonal projections that give the even and odd parts, respectively, of a function by $P_e, P_o :\ L^2_{\mathbb{C}}(\Omega) \to L^2_{\mathbb{C}}(\Omega)$, whose definitions are

$$P_e u(x) := \frac{u(x)+u(-x)}{2}, \quad P_o u(x) := \frac{u(x)-u(-x)}{2}. \tag{4.1}$$

We sketch the derivation of a simple convolution that satisfies homogeneous Dirichlet BC. It is easy to see that for any kernel function C, we have

$$(\mathcal{C} *_{\mathtt{p}} u)(1) = (\mathcal{C} *_{\mathtt{p}} u)(-1).$$

In addition, if C satisfies (1.2), i.e., is even, then $(\mathcal{C} *_{\mathtt{p}} u)(1) = 0$ when u is odd. Likewise, for the antiperiodic case,

$$(\mathcal{C} *_{\mathtt{a}} u)(1) = -(\mathcal{C} *_{\mathtt{a}} u)(-1),$$

holds for any C. Once again, using (1.2), we can easily notice that $(\mathcal{C} *_{\mathtt{a}} u)(1) = 0$ when u is even. Since homogeneous Dirichlet BC are satisfied by $\mathcal{C} *_{\mathtt{p}} P_o$ and $\mathcal{C} *_{\mathtt{a}} P_e$, this suggests that these are functions of the classical operator $A_{\mathtt{D}}$ with homogeneous Dirichlet BC. For the Neumann BC, the situation is similar. By a change of basis, we will prove that $\mathcal{C} *_{\mathtt{p}} P_e$ and $\mathcal{C} *_{\mathtt{a}} P_o$ are both functions of the classical operator $A_{\mathtt{N}}$ and hence automatically satisfy homogeneous Neumann BC. Likewise, $\mathcal{C} *_{\mathtt{p}} P_o$ and $\mathcal{C} *_{\mathtt{a}} P_e$ are both functions of the classical operator $A_{\mathtt{D}}$ and hence satisfy homogeneous Dirichlet BC. Next, we present this change of basis construction.

The PD related governing operator is given by

$$\mathcal{L}u(x) := cu(x) - \int_\Omega \chi_\delta(x-y)u(y)dy, \quad x \in \Omega,$$

where

$$\chi_\delta(x) = \begin{cases} 1, & |x| < \delta, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\hat{\chi}_{\delta,\mathtt{p}}(x)$ and $\hat{\chi}_{\delta,\mathtt{a}}(x)$ denote the periodic and antiperiodic extensions of $\hat{\chi}_\delta(x),\ x \in (-1,1)$, respectively, to $x \in (-2,2)$. It was shown in [3] that the operator

$$\mathcal{M}_{\mathtt{N}} u(x) := cu(x) - \int_\Omega \hat{\chi}_{\delta,\mathtt{p}}(x-y) P_e u(y)dy - \int_\Omega \hat{\chi}_{\delta,\mathtt{a}}(x-y) P_o u(y)dy,$$

is equal to the operator $\mathcal{L}$ in the bulk, i.e., $x \in (-1+\delta, 1-\delta)$. Furthermore, $\mathcal{M}_{\mathtt{N}}$ satisfies the homogeneous Neumann BC. In summary, when $C(x) = \chi_\delta(x)$, we have

$$\mathcal{M}_{\mathtt{N}} = (c - \mathcal{C} *_{\mathtt{p}} P_e - \mathcal{C} *_{\mathtt{a}} P_o). \tag{4.2}$$

In order to carry out numerical experiments for the Neumann BC, we utilize the operator $\mathcal{M}_{\mathtt{N}}$ as the governing operator; see Table 5.1.

In a similar fashion, it was shown in [3] that the operator

$$\mathcal{M}_{\mathtt{D}} u(x) := cu(x) - \int_\Omega \hat{\chi}_{\delta,\mathtt{a}}(x-y) P_e u(y)dy - \int_\Omega \hat{\chi}_{\delta,\mathtt{p}}(x-y) P_o u(y)dy,$$

is equal to the operator $\mathcal{L}$ in the bulk. Furthermore, $\mathcal{L}$ satisfies the homogeneous Dirichlet BC. For $C(x) = \chi_\delta(x)$, we have

$$\mathcal{M}_{\mathtt{D}} = (c - \mathcal{C} *_{\mathtt{a}} P_e - \mathcal{C} *_{\mathtt{p}} P_o). \tag{4.3}$$

In order to carry out numerical experiments for the Dirichlet BC, we utilize the operator $\mathcal{M}_{\mathtt{D}}$ as the governing operator; see Table 5.1. For other related governing operators, see [2].

4.1. **Simple Convolution Operators Satisfying Neumann Boundary Conditions.** We prove that $\mathcal{C} *_{\mathtt{p}} P_e$ and $\mathcal{C} *_{\mathtt{a}} P_o$ are functions of $A_{\mathtt{N}}$. Since we use a change of basis, we recall that the corresponding eigenfunctions for $k \in \mathbb{Z}$ are

$$e^{\mathtt{p}}_k(x) = \frac{1}{\sqrt{2}}\, e^{i\pi k x}, \quad e^{\mathtt{a}}_k(x) = \frac{1}{\sqrt{2}}\, e^{i\pi\left(k+\frac{1}{2}\right)x}.$$

We focus on the operator $\mathcal{C} *_{\mathtt{p}} P_e$. First, we note for even u and $k \in \mathbb{N}^*$ that

$$\langle e^{\mathtt{p}}_k | C\rangle = \langle e^{\mathtt{p}}_{-k} | C\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} \cos(\pi k y) C(y)\, dy,$$

$$\langle e^{\mathtt{p}}_k | u\rangle = \langle e^{\mathtt{p}}_{-k} | u\rangle = \frac{(-1)^k}{\sqrt{2}}\, \langle e^{\mathtt{N}}_{2k} | u\rangle\,.$$

As a consequence, for $k \in \mathbb{N}^*$

$$\begin{aligned} \langle e^{\mathtt{p}}_k|C\rangle\, \langle e^{\mathtt{p}}_k|u\rangle\, e^{\mathtt{p}}_k(x) + \langle e^{\mathtt{p}}_{-k}|C\rangle\, \langle e^{\mathtt{p}}_{-k}|u\rangle\, e^{\mathtt{p}}_{-k}(x) &= \sqrt{2}\,(-1)^k\, \langle e^{\mathtt{p}}_k|C\rangle\, \langle e^{\mathtt{p}}_k|u\rangle\, e^{\mathtt{N}}_{2k}(x) \\ &= \langle e^{\mathtt{p}}_k|C\rangle\, \langle e^{\mathtt{N}}_{2k}|u\rangle\, e^{\mathtt{N}}_{2k}(x). \end{aligned}$$

Hence,

$$\mathcal{C} *_{\mathtt{p}} u = \sum_{k\in\mathbb{Z}} \langle e^{\mathtt{p}}_k|C\rangle\, \langle e^{\mathtt{p}}_k|u\rangle\, e^{\mathtt{p}}_k = \sum_{k=0}^{\infty} \langle e^{\mathtt{p}}_k|C\rangle\, \langle e^{\mathtt{N}}_{2k}|u\rangle\, e^{\mathtt{N}}_{2k} = \sum_{k=0}^{\infty} f_1(k^2)\, \langle e^{\mathtt{N}}_k|u\rangle\, e^{\mathtt{N}}_k,$$

where $f_1 \in B(\sigma(A_{\mathtt{N}}), \mathbb{C})$ is defined by

$$f_1(k^2) := \begin{cases} 0 & \text{if } k \in \mathbb{N} \text{ is odd}, \\ \langle e^{\mathtt{p}}_{k/2}|C\rangle & \text{if } k \in \mathbb{N} \text{ is even}. \end{cases} \tag{4.4}$$

Finally, we see that $\mathcal{C} *_{\mathtt{p}} P_e$ is a function of $A_{\mathtt{N}}$, more precisely

$$\mathcal{C} *_{\mathtt{p}} P_e = f_1(A_{\mathtt{N}}).$$

We now prove that $\mathcal{C} *_a P_o$ is also a function of $A_{\mathtt{N}}$. First, we note for odd u and $k \in \mathbb{N}$ that

$$\langle e^{\mathtt{a}}_k | C\rangle = \langle e^{\mathtt{a}}_{-k-1} | C\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} \cos\left[\pi\big(k+\frac{1}{2}\big)y\right] C(y)\, dy,$$
$$\langle e^{\mathtt{a}}_k | u\rangle = -\langle e^{\mathtt{a}}_{-k-1} | u\rangle = \frac{(-1)^k\, i}{\sqrt{2}} \langle e^{\mathtt{N}}_{2k+1} | u\rangle .$$

As a consequence,

$$\langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{a}}_k | u\rangle\, e^{\mathtt{a}}_k(x) + \langle e^{\mathtt{a}}_{-k-1} | C\rangle \langle e^{\mathtt{a}}_{-k-1} | u\rangle\, e^{\mathtt{a}}_{-k-1}(x) = \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{N}}_{2k+1} | u\rangle\, e^{\mathtt{N}}_{2k+1}(x).$$

Hence,

$$\mathcal{C} *_{\mathtt{a}} u = \sum_{k\in\mathbb{Z}} \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{a}}_k | u\rangle\, e^{\mathtt{a}}_k = \sum_{k=0}^{\infty} \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{N}}_{2k+1} | u\rangle\, e^{\mathtt{N}}_{2k+1} = \sum_{k=0}^{\infty} f_2(k^2) \langle e^{\mathtt{N}}_k | u\rangle\, e^{\mathtt{N}}_k,$$

where $f_2 \in B(\sigma(A_{\mathtt{N}}), \mathbb{C})$ is defined by

$$f_2(k^2) := \begin{cases} \langle e^{\mathtt{a}}_{(k-1)/2} | C\rangle & \text{if } k \in \mathbb{N}^* \text{ is odd,} \\ 0 & \text{if } k \in \mathbb{N}^* \text{ is even.} \end{cases} \tag{4.5}$$

Finally, we see that $\mathcal{C} *_{\mathtt{a}} P_o$ is a function of $A_{\mathtt{N}}$, more precisely

$$\mathcal{C} *_{\mathtt{a}} P_o = f_2(A_{\mathtt{N}}).$$

4.2. **Simple Convolution Operators Satisfying Dirichlet Boundary Conditions.** We prove that $\mathcal{C} *_{\mathtt{p}} P_o$ and $\mathcal{C} *_{\mathtt{a}} P_e$ are functions of $A_{\mathtt{D}}$. We focus on the $\mathcal{C} *_{\mathtt{p}} P_o$ operator. First, we note for odd u and $k \in \mathbb{N}^*$ that

$$\langle e^{\mathtt{p}}_k | C\rangle = \langle e^{\mathtt{p}}_{-k} | C\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} \cos(\pi k y) C(y)\, dy\,,$$
$$\langle e^{\mathtt{p}}_k | u\rangle = -\langle e^{\mathtt{p}}_{-k} | u\rangle = \frac{(-1)^{k+1} i}{\sqrt{2}} \langle e^{\mathtt{D}}_{2k} | u\rangle .$$

As a consequence, for $k \in \mathbb{N}^*$

$$\begin{aligned} \langle e^{\mathtt{p}}_k | C\rangle \langle e^{\mathtt{p}}_k | u\rangle\, e^{\mathtt{p}}_k(x) + \langle e^{\mathtt{p}}_{-k} | C\rangle \langle e^{\mathtt{p}}_{-k} | u\rangle\, e^{\mathtt{p}}_{-k}(x) &= i\sqrt{2}\,(-1)^k \langle e^{\mathtt{p}}_k | C\rangle \langle e^{\mathtt{p}}_k | u\rangle\, e^{\mathtt{D}}_{2k}(x) \\ &= \langle e^{\mathtt{p}}_k | C\rangle \langle e^{\mathtt{D}}_{2k} | u\rangle\, e^{\mathtt{D}}_{2k}(x). \end{aligned}$$

Hence,

$$\mathcal{C} *_{\mathtt{p}} u = \sum_{k\in\mathbb{Z}} \langle e^{\mathtt{p}}_k | C\rangle \langle e^{\mathtt{p}}_k | u\rangle\, e^{\mathtt{p}}_k = \sum_{k=1}^{\infty} \langle e^{\mathtt{p}}_k | C\rangle \langle e^{\mathtt{D}}_{2k} | u\rangle\, e^{\mathtt{D}}_{2k} = \sum_{k=1}^{\infty} f_{1|_{\mathbb{N}^*}}(k^2) \langle e^{\mathtt{D}}_k | u\rangle\, e^{\mathtt{D}}_k,$$

where $f_{1|_{\mathbb{N}^*}}$ is the restriction of f_1 defined in (4.4) to $\mathbb{N}^*$. Finally, we see that $\mathcal{C} *_{\mathtt{p}} P_o$ is a function of $A_{\mathtt{D}}$, more precisely

$$\mathcal{C} *_{\mathtt{p}} P_o = f_1(A_{\mathtt{D}}).$$

We now prove that $\mathcal{C} *_a P_e$ is also a function of $A_{\mathtt{D}}$. First, we note for even u and $k \in \mathbb{N}$ that

$$\langle e^{\mathtt{a}}_k | C\rangle = \langle e^{\mathtt{a}}_{-k-1} | C\rangle = \frac{(-1)^k}{\sqrt{2}} \langle e^{\mathtt{D}}_{2k+1} | C\rangle\,, \langle e^{\mathtt{a}}_k | u\rangle = \langle e^{\mathtt{a}}_{-k-1} | u\rangle = \frac{(-1)^k}{\sqrt{2}} \langle e^{\mathtt{D}}_{2k+1} | u\rangle\,.$$

As a consequence,

$$\begin{aligned} \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{a}}_k | u\rangle\, e^{\mathtt{a}}_k(x) + \langle e^{\mathtt{a}}_{-k-1} | C\rangle \langle e^{\mathtt{a}}_{-k-1} | u\rangle\, e^{\mathtt{a}}_{-k-1}(x) &= \sqrt{2}\,(-1)^k \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{a}}_k | u\rangle\, e^{\mathtt{D}}_{2k+1}(x) \\ &= \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{D}}_{2k+1} | u\rangle\, e^{\mathtt{D}}_{2k+1}(x). \end{aligned}$$

Hence,

$$\mathcal{C} *_{\mathtt{a}} u = \sum_{k\in\mathbb{Z}} \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{a}}_k | u\rangle\, e^{\mathtt{a}}_k = \sum_{k=0}^{\infty} \langle e^{\mathtt{a}}_k | C\rangle \langle e^{\mathtt{D}}_{2k+1} | u\rangle\, e^{\mathtt{D}}_{2k+1} = \sum_{k=1}^{\infty} f_2(k^2) \langle e^{\mathtt{D}}_k | u\rangle\, e^{\mathtt{D}}_k,$$

where f_2 is defined in (4.5). Finally, we see that $\mathcal{C} *_{\mathtt{a}} P_e$ is a function of $A_{\mathtt{D}}$, more precisely

$$\mathcal{C} *_{\mathtt{a}} P_e = f_2(A_{\mathtt{D}}).$$

5. Numerical Experiments

Recalling the governing equation (1.3), we numerically solve the following nonlocal equation

$$u_{tt}(x,t) + f_{\mathtt{BC}}(A_{\mathtt{BC}})u(x,t) = b(x,t), \quad (x,t) \in \Omega \times J, \tag{5.1a}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{5.1b}$$

$$u_t(x,0) = v_0(x) \quad x \in \Omega, \tag{5.1c}$$

where $J = (0,T)$ is some finite time interval, $\Omega = (-1,1)$, b is a given source term, and u_0 and v_0 are given initial conditions. The choice of the subscript $\mathtt{BC} \in \{\mathtt{p},\mathtt{a},\mathtt{N},\mathtt{D}\}$ is determined by the BC that are to be satisfied at the boundary of the physical domain $(-1,1)$. This, in turn, determines the function of the classical operator $f_{\mathtt{BC}}(A_{\mathtt{BC}})$ as described in Table 5.1 where we have defined

$$c := \frac{1}{\sqrt{2}} \int_{-1}^{1} C(y)\, dy. \tag{5.2}$$

Since integral representation of abstract convolutions $\mathcal{C}*_{\mathtt{BC}}$ is more convenient for implementation, these representations are given for $\mathtt{BC} = \mathtt{p}$ and $\mathtt{BC} = \mathtt{a}$ in (3.3) and (3.5), respectively. Furthermore, derived from periodic and antiperiodic cases, the governing operators for the Neumann and Dirichlet BC are given in (4.2) and (4.3), respectively. All governing operators are listed in Table 5.1.

In our theoretical development, note that we have considered homogeneous equations so far because the construction is based exclusively on the operator, not the right hand side function. However, as can be seen from (5.1a), we also consider inhomogeneous equations in this section. For details of the theoretical construction for the inhomogeneous equation, see the foundation paper [1].

5.1. Discretization in Space.

To approximate the solution of (5.1) we begin with discretizing the domain Ω into N subintervals by defining $\Omega_h = \{K_1, K_2, \ldots, K_N\}$ where $K_i = (x_{i-1}, x_i)$ with $-1 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$. We let $h_i = |K_i| = x_i - x_{i-1}$ for $i = 1,\ldots,N$. Given a polynomial degree $\ell \geq 0$, we wish to approximate the solution $u(x,t)$ of (5.1) for a fixed t in the finite element space

$$V_h = \{v \in L^2(\Omega) : v|_K \in P_\ell(K) \text{ for all } K \in \Omega_h\}$$

where $P_\ell(K)$ is the space of polynomials of degree at most ℓ on K.

We define the L^2-inner product on an element $K \in \Omega_h$ as

$$(u,v)_K = \int_K u(x)v(x)\, dx \quad \text{and set} \quad (u,v)_{\Omega_h} = \sum_{K\in\Omega_h} (u,v)_K.$$

BC	$f_{\mathtt{BC}}(A_{\mathtt{BC}})u(x,t)$	BC enforced
p	$(c - \mathcal{C}*_{\mathtt{p}})u(x,t)$	$u(-1,t) = u(1,t), \quad u_x(-1,t) = u_x(1,t)$
a	$(c - \mathcal{C}*_{\mathtt{a}})u(x,t)$	$u(-1,t) = -u(1,t),\ u_x(-1,t) = -u_x(1,t)$
N	$\sqrt{2}\,[(c - \mathcal{C} *_{\mathtt{p}} P_e - \mathcal{C} *_{\mathtt{a}} P_o)]u(x,t)$	$u_x(-1,t) = u_x(1,t) = 0$
D	$\sqrt{2}\,[(c - \mathcal{C} *_{\mathtt{p}} P_o - \mathcal{C} *_{\mathtt{a}} P_e)]u(x,t)$	$u(-1,t) = u(1,t) = 0$

Table 5.1. The choice of nonlocal operators based on the boundary conditions enforced.

For approximation of (5.1), we use a Galerkin projection as used in [8, 9] and consider the following (semidiscrete) approximation: Find $u^h : J \times V_h \to \mathbb{R}$ such that

$$(u^h_{tt}, v)_{\Omega_h} + (f_{\mathtt{BC}}(A_{\mathtt{BC}})u^h, v)_{\Omega_h} = (b, v)_{\Omega_h} \quad \text{for all } v \in V_h, \tag{5.3a}$$

$$u^h|_{t=0} = \Pi_h u_0, \tag{5.3b}$$

$$u^h_t|_{t=0} = \Pi_h v_0. \tag{5.3c}$$

Here, Π_h denotes the L^2-projection onto V_h.

5.2. **Discretization in Time.** The discretization of (5.1) by the Galerkin method (5.3) leads to the second-order system of ordinary differential equations

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}^h(t) = \mathbf{b}^h(t), \qquad t \in J, \tag{5.4}$$

with initial conditions

$$\mathbf{M}\mathbf{u}^h(0) = \mathbf{u}^h_0, \qquad \mathbf{M}\dot{\mathbf{u}}^h = \mathbf{v}^h_0. \tag{5.5}$$

Here, $\mathbf{M}$ denotes the mass matrix and $\mathbf{A}$ denotes the stiffness matrix. To discretize (5.4)-(5.5) in time, we employ the Newmark time-stepping scheme as described in [21].

Let k denote the time step and set $t_n = n \cdot k$ for $n = 1, 2, \dots$. The Newmark scheme we employ consists of finding approximations $\{\mathbf{u}^h_n\}_n$ to $\mathbf{u}^h(t_n)$ such that

$$\begin{aligned} \mathbf{M}\mathbf{u}^h_1 &= \big(\mathbf{M} - \frac{1}{2}k^2\mathbf{A}\big)\mathbf{u}^h_0 + k\mathbf{M}\mathbf{v}^h_0 + \frac{1}{2}k^2\mathbf{b}^h_0, \\ \mathbf{M}\mathbf{u}^h_{n+1} &= \big(2\mathbf{M} - k^2\mathbf{A}\big)\mathbf{u}^h_n - \mathbf{M}\mathbf{u}^h_{n-1} + k^2\mathbf{b}^h_n, \end{aligned}$$

for $n = 1, 2, \dots, N_t - 1$ where $N_t k = T$, and $\mathbf{b}_n = \mathbf{b}(t_n)$. Although there is a more general version of the Newmark time-stepping scheme, we made this particular choice due to the fact that it is second-order accurate and is explicit in the sense that at each time step we only have to solve a linear system with a coefficient matrix $\mathbf{M}$ that is block diagonal. Hence, $\mathbf{M}$ can be inverted at a very low computational cost. For other Newmark schemes the coefficient matrix of the linear system would be $\mathbf{M} + k^2\beta\mathbf{A}$ for some $\beta > 0$ which needs to be inverted at each time step. For a detailed discussion of more general Newmark time integration schemes we refer to [21].

5.3. **Implementation Details.** Let us describe a few details regarding the computation of the stiffness matrix $\mathbf{A}$. Let $K \in \Omega_h$ and let $\{\phi^K_j : j = 1, \dots, \ell + 1\}$ be a basis for $P_\ell(K)$. To fix ideas, let us consider the case $\mathtt{BC} = \mathtt{p}$ so that

$$f_{\mathtt{BC}}(A_{\mathtt{BC}})u(x, t) = (c - \mathcal{C}*_{\mathtt{p}})u(x, t).$$

The remaining cases $\mathtt{BC} = \mathtt{a}, \mathtt{N}, \mathtt{D}$ are similar.

First of all, we need to compute the constant c in (5.2). In the cases where C is an elementary function such as a (piecewise) polynomial, the exact value of this constant can be computed by direct integration. However, in the general case, we have to resort to numerical quadrature. We simply compute

$$c = \frac{1}{\sqrt{2}} \sum_{K \in \Omega_h} \int_K C(x)dx$$

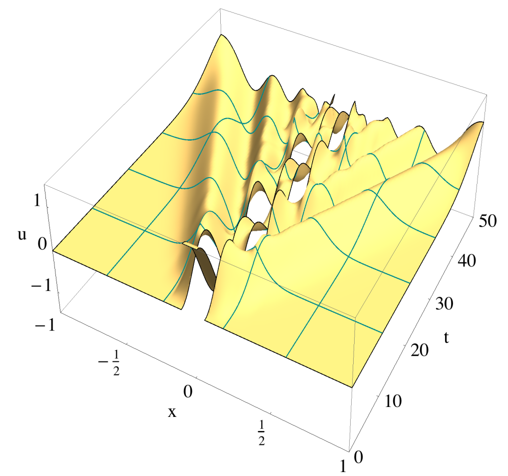
where the integral on each element $K \in \Omega_h$ is approximated by a quadrature rule. In this case, if C happens to have discontinuities or kinks in Ω, in order to obtain an accurate approximation to c, we have to ensure that the nodes of the discrete domain Ω_h are aligned with these discontinuities.

The matrix $\mathbf{A}$ is of size $N(\ell+1)\times N(\ell+1)$ and has a block structure. Each block-row of size $(\ell+1)\times N(\ell+1)$ corresponding to an element $K \in \Omega_h$ is determined by the equations

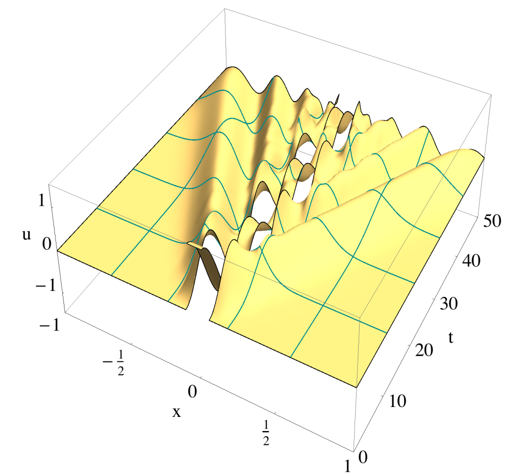
$$(f_{\mathtt{p}}(A_{\mathtt{p}})u^h, \phi^K_i)_K = (b, \phi^K_i)_K, \qquad \text{for } i = 1, 2, \dots, \ell + 1.$$

Inserting the definition of $f_{\mathtt{p}}(A_{\mathtt{p}})$, we get

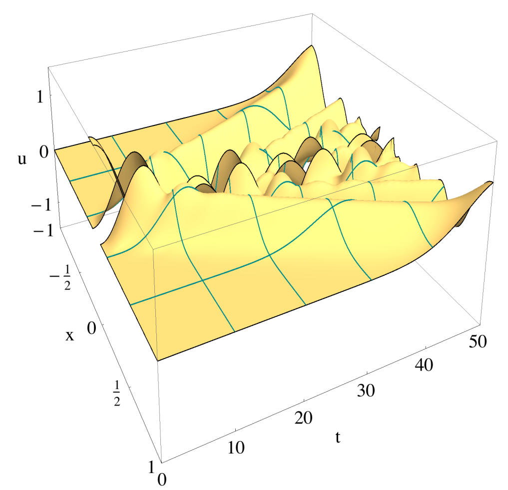
$$\begin{aligned} (f_{\mathtt{p}}(A_{\mathtt{p}})u^h, \phi^K_i)_K &= ((c - \mathcal{C}*_{\mathtt{p}})u^h, \phi^K_i)_K \\ &= c(u^h, \phi^K_i)_K - (\mathcal{C} *_{\mathtt{p}} u^h, \phi^K_i)_K. \end{aligned}$$



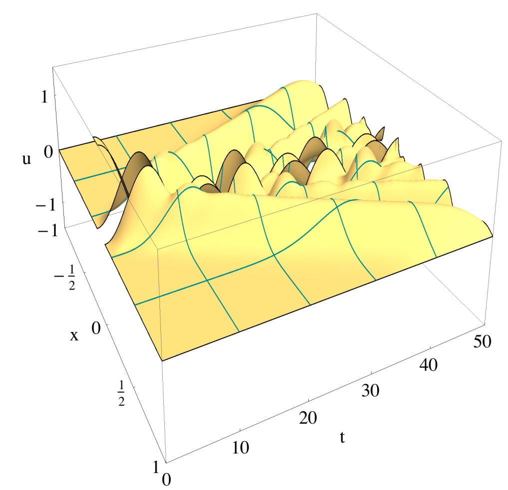
(a) Solution u with initial data $u(x,0) = u_{0,\mathrm{disc}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



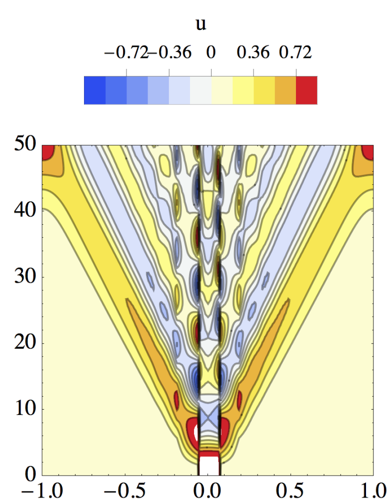
(b) Solution u with initial data $u(x,0) = u_{0,\mathrm{disc}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



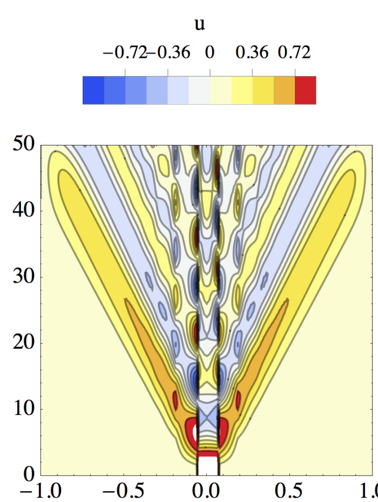
(c) The same solution from Fig. 5.1(a) from a boundary point of view.



(d) The same solution from Fig. 5.1(b) from a boundary point of view.

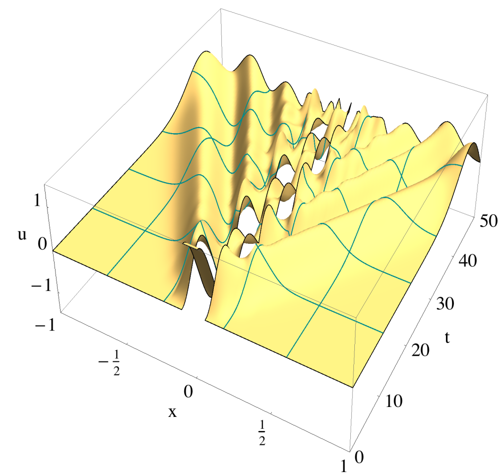


(e) Contour plot of u from Fig. 5.1(a).

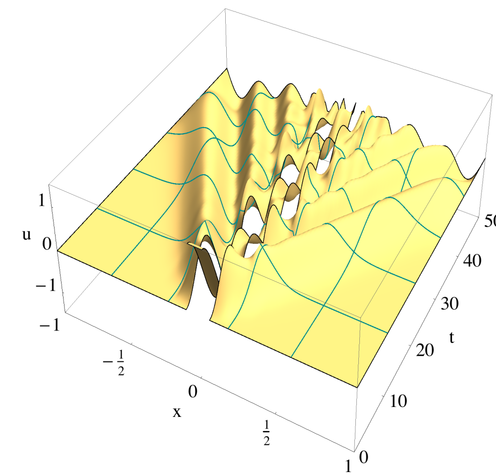


(f) Contour plot of u from Fig. 5.1(b).

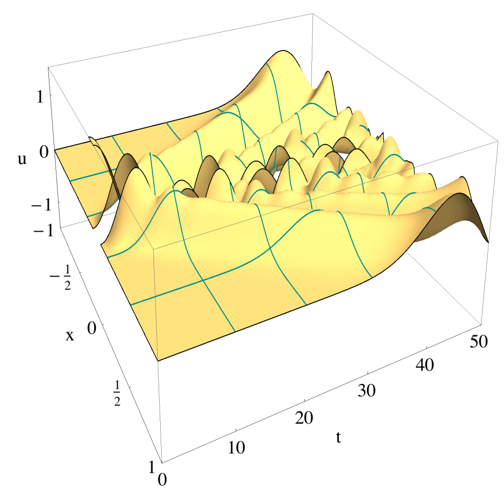
FIGURE 5.1. Solution to the nonlocal wave equation with periodic (left) and antiperiodic (right) boundary conditions, discontinuous initial displacement, and vanishing initial velocity.



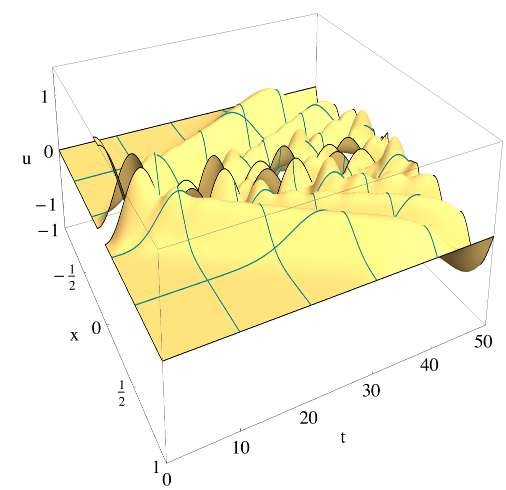
(a) Solution u with initial data $u(x,0) = u_{0,\mathrm{disc}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



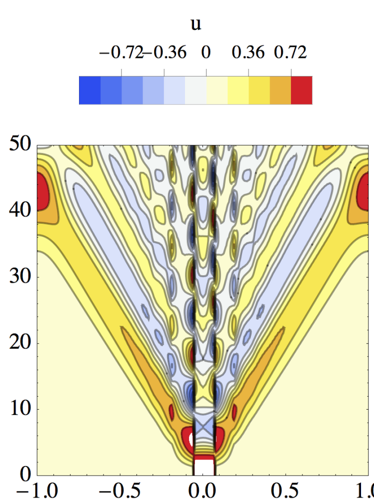
(b) Solution u with initial data $u(x,0) = u_{0,\mathrm{disc}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



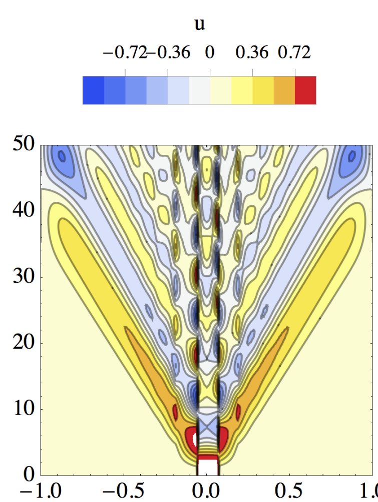
(c) The same solution from Fig. 5.2(a) from a boundary point of view.



(d) The same solution from Fig. 5.2(b) from a boundary point of view.

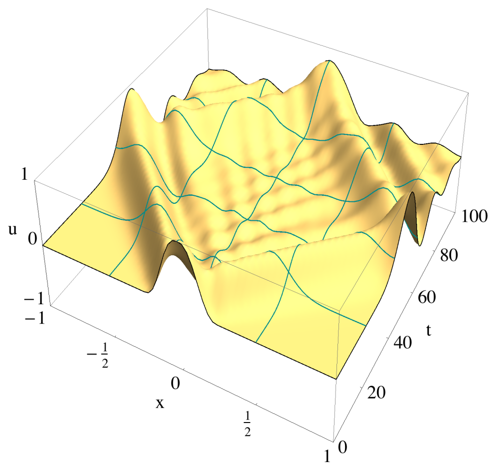


(e) Contour plot of u from Fig. 5.2(a).

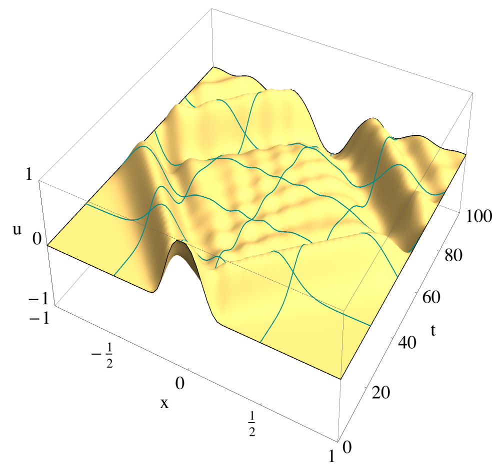


(f) Contour plot of u from Fig. 5.2(b).

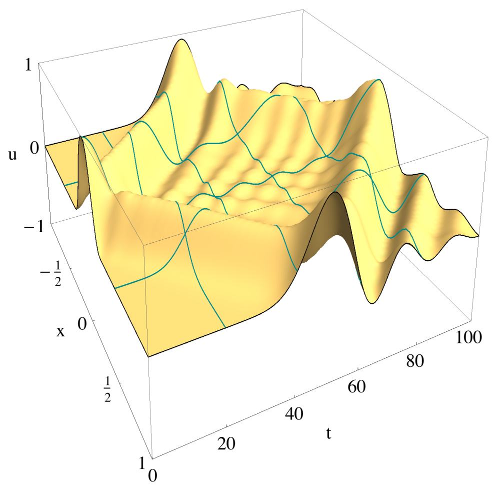
FIGURE 5.2. Solution to the nonlocal wave equation with Neumann (left) and Dirichlet (right) boundary conditions, discontinuous initial displacement, and vanishing initial velocity.



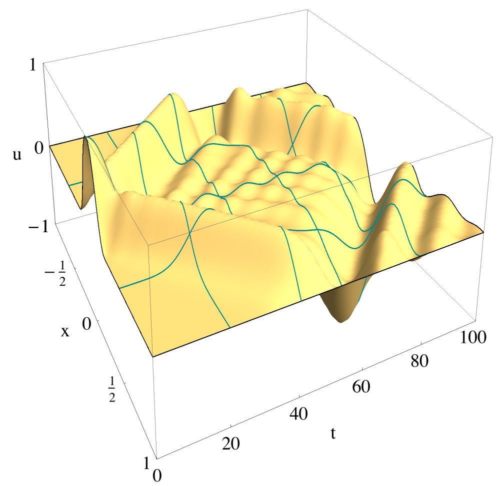
(a) Solution u with initial data $u(x,0) = u_{0,\text{cont}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



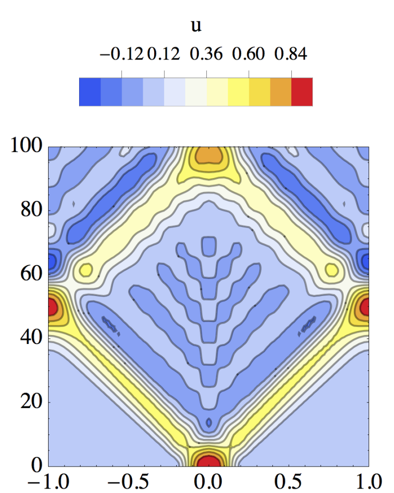
(b) Solution u with initial data $u(x,0) = u_{0,\text{disc}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



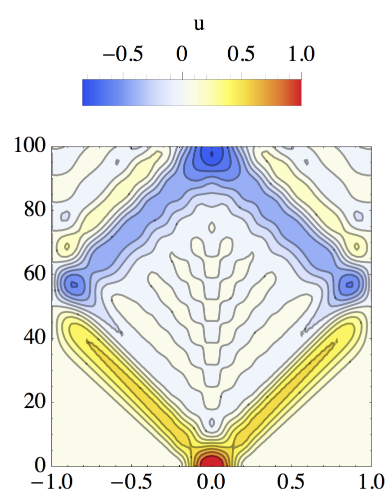
(c) The same solution from Fig. 5.3(a) from a boundary point of view.



(d) The same solution from Fig. 5.3(b) from a boundary point of view.

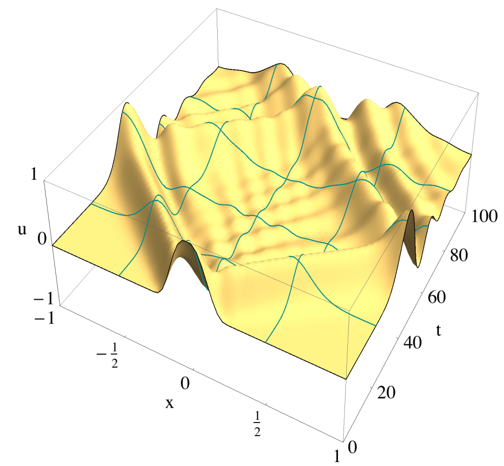


(e) Contour plot of u from Fig. 5.3(a).

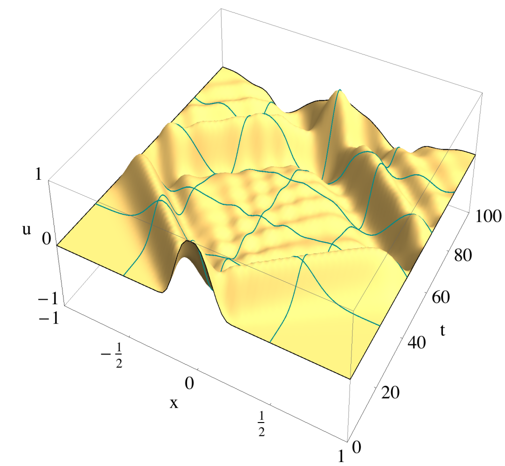


(f) Contour plot of u from Fig. 5.3(b).

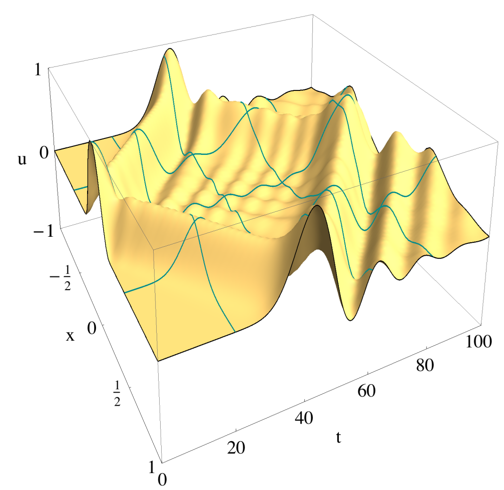
FIGURE 5.3. Solution to the nonlocal wave equation with periodic (left) and antiperiodic (right) boundary conditions, continuous initial displacement, and vanishing initial velocity.



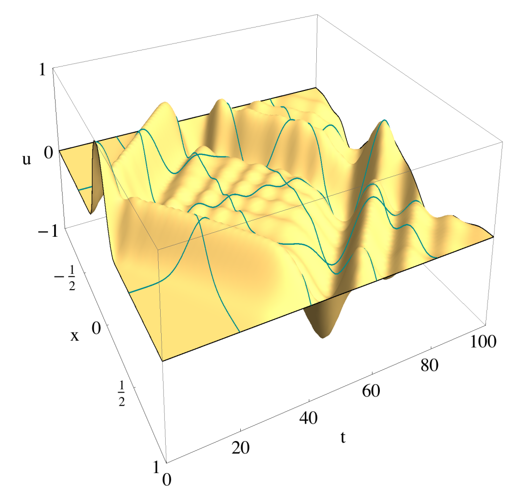
(a) Solution u with initial data $u(x,0) = u_{0,\mathrm{disc}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



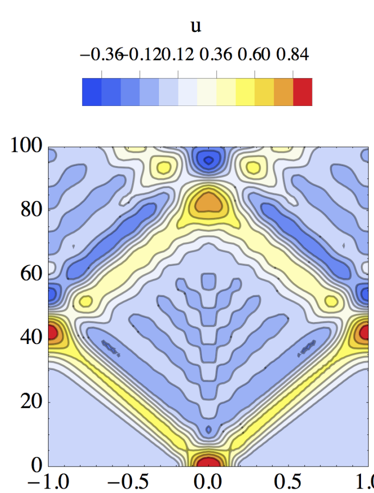
(b) Solution u with initial data $u(x,0) = u_{0,\mathrm{cont}}(x)$ and $u_t(x,0) = 0,\ x \in (-1,1)$. Initial data view.



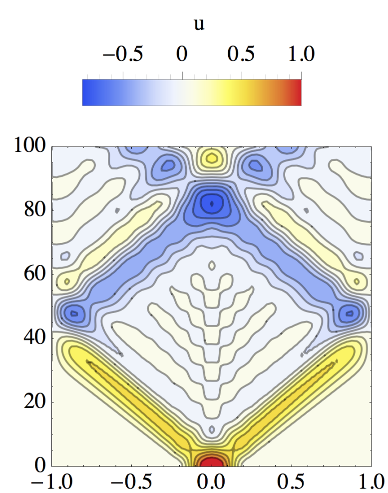
(c) The same solution from Fig. 5.4(a) from a boundary point of view.



(d) The same solution from Fig. 5.4(b) from a boundary point of view.

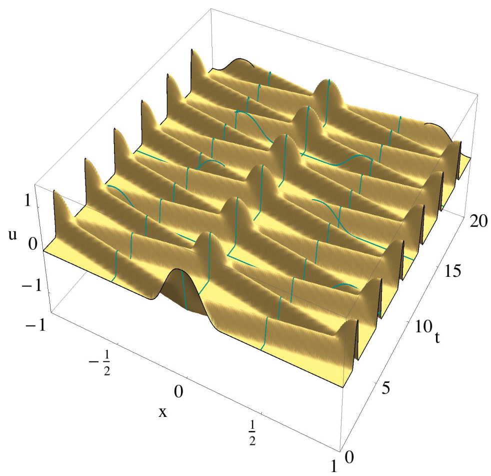


(e) Contour plot of u from Fig. 5.4(a).

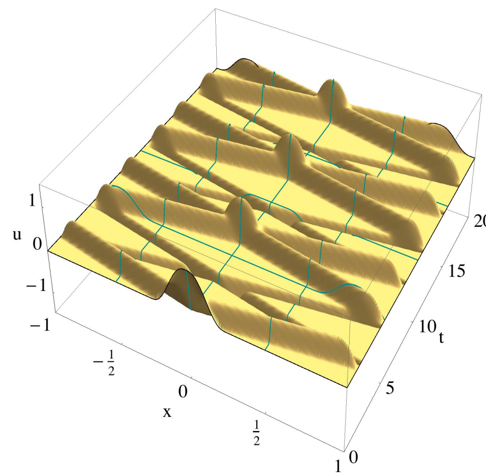


(f) Contour plot of u from Fig. 5.4(b).

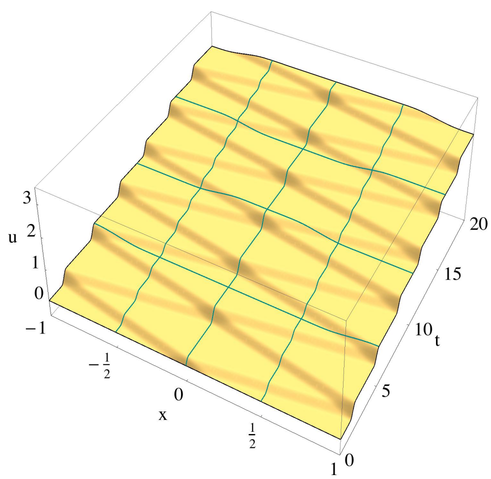
FIGURE 5.4. Solution to the nonlocal wave equation with Neumann (left) and Dirichlet (right) boundary conditions, continuous initial displacement, and vanishing initial velocity.



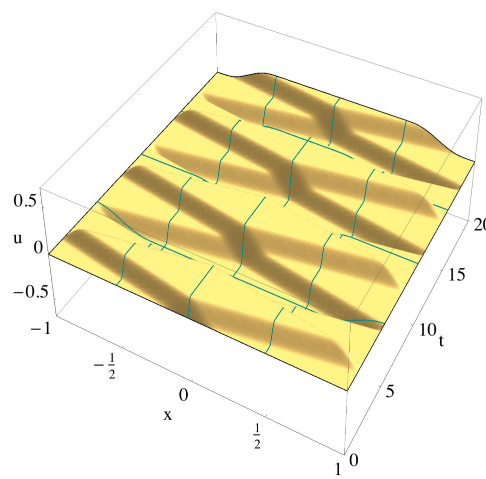
(a) Solution u to the classical (local) wave equation with initial data $u(x,0) = u_{0,\text{cont}}(x)$ defined in (5.7) and $u_t(x,0) = 0$.



(b) Solution u to the classical (local) wave equation with initial data $u(x,0) = u_{0,\text{cont}}(x)$ defined in (5.7) and $u_t(x,0) = 0$.



(c) Solution u to the classical (local) wave equation with initial data $u(x,0) = 0$ and $u_t(x,0) = u_{0,\text{cont}}(x)$ defined in (5.7).



(d) Solution u to the classical (local) wave equation with initial data $u(x,0) = 0$ and $u_t(x,0) = u_{0,\text{cont}}(x)$ defined in (5.7).

FIGURE 5.5. Solution to the classical wave equation with Neumann ((a) and (c)) and Dirichlet ((b) and (d)) boundary conditions with vanishing initial velocity ((a) and (b)) and vanishing initial displacement ((c) and (d)).

The computation of the first term is standard, but we would like elaborate on a few details regarding the computation of the second term. At any fixed time $t \in J$ and for a fixed element $T \in \Omega_h$, we have the restriction, $u^{h,T}$, of u^h on T has the expansion

$$u^{h,T}(x,t) = \sum_{j=1}^{\ell+1} u_j^T(t)\phi_j^T(x).$$

Then, since

$$\begin{aligned}\mathcal{C} *_{\mathtt{p}} u^h(x,t) &= \int_{-1}^{1} \widehat{C}_{\mathtt{p}}(x-y)\, u^h(y,t)\, dy \\ &= \sum_{T\in\Omega_h} \sum_{j=1}^{\ell+1} u_j^T(t) \int_T \widehat{C}_{\mathtt{p}}(x-y)\phi_j^T(y)\, dy,\end{aligned}$$

we have

$$(\mathcal{C} *_{\mathtt{p}} u^h, \phi_i^K)_K = \sum_{T\in\Omega_h} \sum_{j=1}^{\ell+1} u_j^T(t) \int_K R_j^T(x)\phi_i^K(x)\, dx \tag{5.6}$$

where

$$R_j^T(x) := \int_T \widehat{C}_{\mathtt{p}}(x-y)\phi_j^T(y)\, dy.$$

Thus, we need to compute pointwise values of R_j^T which will be achieved through numerical quadrature. Note that the micromodulus function C may have points of discontinuities or kinks (or higher order derivatives of C may not be continuous) in Ω. Hence, when computing $R_j^T(x)$, we need to take these points into account, for example, when using Gaussian quadrature which requires the smoothness of the integrand for optimal order accuracy. Furthermore, even if C is arbitrarily smooth in Ω, its extension $\widehat{C}_{\mathtt{p}}$ may not be smooth in $[-2,2]$. Since the integrand involves $\widehat{C}_{\mathtt{p}}(x-y)$ which is a translation of $\widehat{C}_{\mathtt{p}}(-y) = \widehat{C}_{\mathtt{p}}(y)$ by x units to the left, we always have to account for possible singularities of $\widehat{C}_{\mathtt{p}}(y)$ at the end points, $\{-1,1\}$, of the domain Ω. Suppose $y_s \in T$ is such that $\widehat{C}_{\mathtt{p}}(x-y_s)$ has a singularity in K. Then the integral defining $R_j^T(x)$ has to be computed by writing $T = T_1 \cup T_2$ where $T = (x_L, x_R)$, $T_1 = (x_L, y_s)$ and $T_2 = (y_s, x_R)$, and applying numerical quadrature on both subintervals. A similar treatment is needed when computing the integral $\int_K R_j^T(x)\phi_j^K(x)\, dx$.

Due to the nonlocal nature of the problem, the stiffness matrix $\mathbf{A}$ is not necessarily sparse. This can be seen from (5.6) by observing that R_j^T does not necessarily vanish on the element K for $T \neq K$. The sparsity structure of $\mathbf{A}$ is determined by the support of the micromodulus function C. More explicitly, the wider the support of C, the less sparse $\mathbf{A}$ is. Symmetry and positive definiteness of the stiffness matrix are the consequences of the self-adjointness and positivity of the governing operator, respectively; see [1]. For the case of periodic and Neumann BC, the stiffness matrix becomes positive semidefinite and these systems can be solved by using numerical methods described in [12, 20]. Finally, we would like to point out that the assembly of the stiffness matrix as well as the mass matrix is independent of the time step and is performed only once.

5.4. **Approximations to Explicitly Known Exact Solutions.** In order to ascertain the convergence performance of the scheme described above, we display some numerical results corresponding to explicitly known exact solutions. We solve one example corresponding to each BC type. We take the exact solution corresponding to each BC as given in Table 5.2 and compute the corresponding right-hand side function $b(x,t)$. Note that, since the operator $f_{\mathtt{BC}}(A_{\mathtt{BC}})$ is different for each BC, the corresponding source term $b(x,t)$ also differs. We take the micromodulus function C to be the unit box on Ω, namely,

$$C(x) = \begin{cases} 1, & x \in [-1/2, 1/2], \\ 0, & \text{otherwise.} \end{cases}$$

For each case, we compute the exact solution until the final time $T = 20$ and compute the relative L^2-error $\left\|(u-u^h)(\cdot,T)\right\|_0 / \left\|u(\cdot,T)\right\|_0$. We first compute an approximate solution with a uniform coarse mesh with $N = 2^3$ elements and then refine the mesh by subdividing each element into two elements of equal size. In each case, as the time step of the Newmark scheme we take $\Delta t = 0.005$ so that the explicit Newmark time integration scheme is stable. In all of our examples, we found out that taking Δt so that $\Delta t < h/10$ is sufficient for stability. Note that since the Newmark scheme is second order accurate in time, and all of the exact solutions in Table 5.2 is of the form $u(x,t) = T(t)X(x)$ with $T(t) = t^2$, a second order polynomial, it is guaranteed that the dominant error is that in the space variable.

We display our numerical results in Table 5.3. Therein, the column labeled ℓ indicates the polynomial degree we used to compute u^h, and the column labeled "mesh" denotes the mesh we used to compute the relative error displayed in the corresponding row, more explicitly, mesh$=i$ means we used a uniform mesh with $N = 2^i$ elements. In the column labeled "order" we display an approximate order of convergence as follows. If e_i denotes the relative error with mesh$=i$, then we display the quantity

$$r_{i+1} = -\frac{1}{\log 2}\log\left(\frac{e_{i+1}}{e_i}\right)$$

at the row corresponding to mesh$=i+1$. The results displayed in Table 5.3 suggest an error estimate of the form

$$\frac{\left\|(u-u^h)(\cdot,T)\right\|_0}{\|u(\cdot,T)\|_0} \leq D\, h^{\ell+1}$$

for some constant D independent of u and h, that is, the method converges *optimally* with respect to the mesh size.

5.5. **Approximations to Unknown Solutions.** Here we display some numerical results in which we solve (5.3) with $b = 0$ on $\Omega \times J$. In this case, we do not have an explicit representation of the solution and merely rely on numerical computing. We consider two initial displacement functions

BC	$u(x,t)$
p	$t^2(\sin(\pi x)+\cos(\pi x))$
a	$t^2(x^4-1)$
N	$t^2((x^2-1)^2-8/15)$
D	$t^2(1+\sin(\pi x)+\cos(\pi x))$

TABLE 5.2. Known exact solutions used in numerical experiments.

		periodic		antiperiodic		Neumann		Dirichlet	
ℓ	mesh	error	order	error	order	error	order	error	order
0	3	2.32E-01	–	1.53E-01	–	2.34E-01	–	1.83E-01	–
	4	1.14E-01	1.02	6.88E-02	1.15	1.15E-01	1.03	8.35E-02	1.13
	5	5.68E-02	1.01	3.29E-02	1.06	5.72E-02	1.01	4.05E-02	1.04
	6	2.84E-02	1.00	1.62E-02	1.02	2.85E-02	1.00	2.01E-02	1.01
	7	1.42E-02	1.00	8.10E-03	1.00	1.43E-02	1.00	1.00E-02	1.00
1	3	2.28E-02	–	1.46E-02	–	2.30E-02	–	1.62E-02	–
	4	5.74E-03	1.99	3.69E-03	1.99	5.91E-03	1.96	4.06E-03	1.99
	5	1.44E-03	2.00	9.25E-04	2.00	1.49E-03	1.99	1.02E-03	2.00
	6	3.59E-04	2.00	2.32E-04	2.00	3.73E-04	2.00	2.54E-04	2.00
	7	8.98E-05	2.00	5.79E-05	2.00	9.32E-05	2.00	6.35E-05	2.00
2	3	1.52E-03	–	8.03E-04	–	2.05E-03	–	1.07E-03	–
	4	1.90E-04	2.99	1.01E-04	2.99	2.47E-04	3.05	1.35E-04	2.99
	5	2.38E-05	3.00	1.26E-05	3.00	3.06E-05	3.01	1.69E-05	3.00
	6	2.98E-06	3.00	1.58E-06	3.00	3.82E-06	3.00	2.11E-06	3.00
	7	3.73E-07	3.00	1.97E-07	3.00	4.77E-07	3.00	2.63E-07	3.00
3	3	7.51E-05	–	2.21E-05	–	5.03E-04	–	5.31E-05	–
	4	4.71E-06	3.99	1.38E-06	4.00	3.16E-05	3.99	3.33E-06	3.99
	5	2.95E-07	4.00	8.62E-08	4.00	1.98E-06	4.00	2.08E-07	4.00
	6	1.84E-08	4.00	5.39E-09	4.00	1.25E-07	3.99	1.30E-08	4.00

TABLE 5.3. History of convergence with known exact solutions for all BC types.

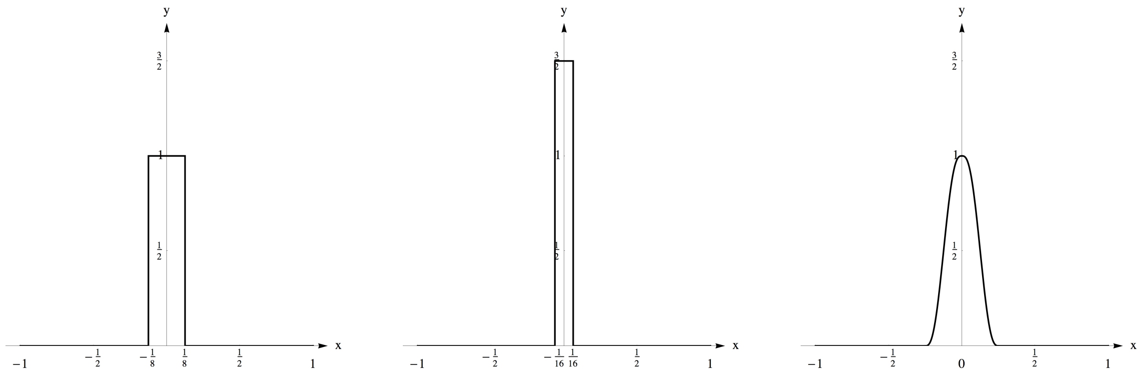


FIGURE 5.6. Micromodulus function $C(x)$ (left). Discontinuous (middle) and continuous (right) initial displacement functions $u_{0,\mathrm{disc}}(x)$ and $u_{0,\mathrm{cont}}(x)$, respectively.

$$u_{0,\mathrm{disc}}\,(x) = \begin{cases} 3/2, & x \in [-1/16, 1/16], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_{0,\mathrm{cont}}(x) = \begin{cases} 0, & x \in (-1, -1/4), \\ (1+4x)^3(96x^2 - 12x + 1), & x \in [-1/4, 0), \\ (1-4x)^3(96x^2 + 12x + 1), & x \in [0, 1/4], \\ 0, & x \in (1/4, 1). \end{cases} \tag{5.7}$$

In all cases, the initial velocity $v_0(x) = 0$ for all $x \in \Omega$. The micromodulus function is taken to be

$$C(x) = \begin{cases} 1, & x \in [-1/8, 1/8], \\ 0, & \text{otherwise}. \end{cases}$$

These functions are displayed in Fig. 5.6. We use the polynomial degree $\ell = 2$ on a mesh with $N = 64$ and $N = 128$ element for continuous and discontinuous initial data, respectively. For each BC case, we depict the associated wave propagation; see Fig. 5.1, 5.2, 5.3, and 5.4.

For $t \in \mathbb{R}$, we have proved that the solution is discontinuous if and only if the initial data is discontinuous; see the foundation paper [1]. Furthermore, the position of discontinuity is determined by the initial data and should remain stationary. Since we use vanishing initial velocity, the explicit solution expression is given by [1]

$$u(x,t) = \cos\left(t\sqrt{c}\right) u(x,0) + u_{\mathrm{smth}}(x,t),$$

where $u_{\mathrm{smth}}(\cdot,t)$ is a continuous function for $t \in \mathbb{R}$. As seen in Fig. 5.1 and 5.2, discontinuities of the initial data remain stationary at $x = -1/16$ and $x = 1/16$. In addition, we observe that away from discontinuities, the solution is smooth, verifying the smoothness of $u_{\mathrm{smth}}(\cdot,t)$.

We also numerically verify that the prescribed BC are satisfied for all $t \in [0, 50]$ and $t \in [0, 100]$ for discontinuous and continuous initial data, respectively. For instance, it is easy to see that homogeneous Dirichlet BC are satisfied in Fig. 5.2 and 5.4. Furthermore, the governing operator preserves the reflection symmetry. In other words, since initial data (both $u_{0,\mathrm{disc}}$ and $u_{0,\mathrm{cont}}$) are symmetric with respect to $x = 0$, the displacement is symmetric with respect to $x = 0$, which can easily be observed by the symmetry in contour plots; for discontinuous initial data see Fig. 5.1(e), 5.1(f), 5.2(e), and 5.2(f) and for continuous initial data, see 5.3(e), 5.3(f), 5.4(e), and 5.4(f).

We also report solutions of classical (local) and nonlocal equations with continuous initial data $u_{0,\mathrm{cont}}(x)$ given in (5.7). First of all, wave separation behavior in the bounded domain is similar to that from the unbounded domain case as reported in [11]. There are two groups of propagating waves, one group towards the left and another towards the right boundary. In the nonlocal case, we observe oscillatory recurrent wave separation and oscillations are located at the center of the initial pulse. Hence, the wave patterns are symmetric with respect to $x = 0$; see Fig. 5.3 and 5.4.

As far as the boundary behavior goes, in nonlocal problems, we observe a similar wave reflection pattern from the boundary as in classical problems. In the classical case, we see that the Neumann and the Dirichlet BC create reflections of same and opposite signs, respectively; for the Neumann BC, see Fig. 5.5(a) and 5.5(c) and for the Dirichlet BC, see Fig. 5.5(b) and 5.5(d). A parallel behavior is observed for the nonlocal Neumann and Dirichlet cases; see Fig. 5.4.

6. Conclusion

The main goal in this paper was to apply the concept of abstract convolution operator to nonlocal problems such as the nonlocal wave equation given in (1.3) and carry out a numerical study of the resulting operators. The choice of the Hilbert basis provides a flexibility in the construction of the abstract convolution operator. We made this construction concrete by choosing the basis to be the eigenbasis of the classical operator with prominent local BC. This is precisely the mechanism we used to incorporate local BC into nonlocal problems. The theoretical aspects and foundations of this construction process are discussed in our foundation paper [1].

In the case of periodic and antiperiodic BC, integral representations of the abstract convolutions are relatively direct to establish. Such representations can also be obtained for the case of Neumann BC, but with considerably more effort exploiting half-wave symmetry. For Dirichlet BC, this integral representation involves an orthogonal projection of the micromodulus function onto a closed subspace defined in terms of the eigenbasis. We give an integral representation of this projection which does not depend on the eigenbasis. Applying convolutions of the periodic and antiperiodic cases, we constructed additional integral convolutions, what we call *simple convolutions*, satisfying Neumann and Dirichlet BC.

Our foundation paper [1] and this paper together present a unique way of combining the powers of abstract operator theory with numerical computing. The abstractness of the methods used in the foundation paper allows generalization to other nonlocal theories. To substantiate the uniqueness of our treatment, we provided a comprehensive numerical study of the solutions of the nonlocal wave equation. We accomplished to demonstrate two crucial goals: For $t \in \mathbb{R}$ and each BC considered, discontinuities of the initial data remain stationary and BC are satisfied by the solution. In the depicted solutions, for discretization, we employed a weak formulation based on a Galerkin projection and used piecewise polynomials on each element which allows discontinuities of the approximate solution at the element borders. We carried out a history of convergence study to ascertain the convergence behavior of the method with respect to the polynomial order and observed optimal convergence.

Our ongoing work aims to extend the novel operators to vector valued problems [4] which will allow the extension of PD to applications that require local BC. Furthermore, we hope that our novel approach potentially will avoid altogether the surface effects seen in PD. We conclude that we added valuable numerical tools to the arsenal of methods to treat nonlocal problems and compute their solutions.

References

[1] Aksoylu, B., Beyer, H.R., Celiker, F.: Theoretical foundations of incorporating local boundary conditions into nonlocal problems. Rep. Math. Phys. (2017). In press.

[2] Aksoylu, B., Celiker, F.: Comparison of nonlocal operators utilizing perturbation analysis. In: B.K. et al. (ed.) Numerical Mathematics and Advanced Applications ENUMATH 2015, *Lecture Notes in Computational Science and Engineering*, vol. 112, pp. 589–606. Springer (2016)

[3] Aksoylu, B., Celiker, F.: Nonlocal problems with local Dirichlet and Neumann boundary conditions. J. Mech. Mater. Struct. **12**(4), 425–437 (2017). DOI 10.2140/jomms.2017.12.425

[4] Aksoylu, B., Celiker, F., Kilicer, O.: Local boundary conditions in vector valued nonlocal problems. In preparation.

[5] Aksoylu, B., Celiker, F., Kilicer, O.: Nonlocal problems with local boundary conditions: An overview. Submitted.

[6] Aksoylu, B., Celiker, F., Kilicer, O.: Nonlocal problems with local boundary conditions in higher dimensions. Submitted.

[7] Aksoylu, B., Mengesha, T.: Results on nonlocal boundary value problems. Numer. Funct. Anal. Optim. **31**(12), 1301–1317 (2010)

[8] Aksoylu, B., Parks, M.L.: Variational theory and domain decomposition for nonlocal problems. Appl. Math. Comp. **217**, 6498–6515 (2011). DOI 10.1016/j.amc.2011.01.027

[9] Aksoylu, B., Unlu, Z.: Conditioning analysis of nonlocal integral operators in fractional Sobolev spaces. SIAM J. Numer. Anal. **52**(2), 653–677 (2014)

[10] Andreu-Vaillo, F., Mazon, J.M., Rossi, J.D., Toledo-Melero, J.: Nonlocal Diffusion Problems, *Mathematical Surveys and Monographs*, vol. 165. American Mathematical Society and Real Socied Matematica Espanola (2010)

[11] Beyer, H.R., Aksoylu, B., Celiker, F.: On a class of nonlocal wave equations from applications. J. Math. Phy. **57**(6), 062902 (2016). DOI 10.1063/1.4953252. Eid: 062902

[12] Bochev, P.B., Lehoucq, R.B.: On the Finite Element Solution of the Pure Neumann Problem. SIAM Review **47**(1), 50–66 (2005)

[13] Du, Q., Gunzburger, M., Lehoucq, R.B., Zhou, K.: Analysis and approximation of nonlocal diffusion problems with volume constraints. SIAM Rev. **54**, 667–696 (2012)

[14] Du, Q., Lipton, R.: Peridynamics, Fracture, and Nonlocal Continuum Models. SIAM News, Volume 47, Number 3, April 2014

[15] Du, Q., Zhou, K.: Mathematical analysis for the peridynamic nonlocal continuum theory. ESAIM: Math. Model. Numer. Anal. **45**, 217–234 (2011)

[16] Emmrich, E., Lehoucq, R.B., Puhst, D.: Peridynamics: a nonlocal continuum theory. In: M. Griebel, M.A. Schweitzer (eds.) Meshfree Methods for Partial Differential Equations VI, vol. 89, pp. 45–65. Springer-Verlag, Berlin, Heidelberg (2013)

[17] Emmrich, E., Weckner, O.: On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity. Commun. Math. Sci. **5**(4), 851–864 (2007)

[18] Emmrich, E., Weckner, O.: The peridynamic equation and its spatial discretization. Math. Model. Anal. **12**(1), 17–27 (2007)

[19] Erbay, H.A., Erkip, A., Muslu, G.M.: The Cauchy problem for a one-dimensional nonlinear elastic peridynamic model. J. Differ. Equ. **252**, 4392–4409 (2012)

[20] Gockenbach, M.S.: Understanding And Implementing the Finite Element Method. Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania (2006)

[21] Grote, M.J., Schneebeli, A., Schötzau, D.: Galerkin finite element method for the wave equation. SIAM Journal on Numerical Analysis **44**(6), 2408–2431 (2006)

[22] Kalthoff, J.F., Winkler, S.: Failure mode transition at high rates of shear loading. In: C. Chiem, H.-D.Kunze, L. Meyer (eds.) Impact Loading and Dynamic Behavior of Materials, vol. 1, pp. 185–195. DGM Informationsgesellschaft Verlag, Berlin (1988)

[23] Kilic, B., Madenci, E.: Prediction of crack paths in a quenched glass plate by using peridynamic theory. Int. J. Fract. **156**, 165–177 (2009)

[24] Lipton, R.: Dynamic brittle fracture as a small horizon limit of peridynamics. J. Elasticity **117**, 21–50 (2014)

[25] Madenci, E., Oterkus, E.: Peridynamic Theory and Its Applications. Springer, New York Heidelberg Dordrecht London (2014). DOI 10.1007/978-1-4614-8465-3

[26] Mengesha, T., Du, Q.: The bond-based peridynamic system with Dirichlet-type volume constraint. Proc. R. Soc. Edinburgh, Sect. A **144A**, 161–186 (2014)

[27] Mengesha, T., Du, Q.: Nonlocal constrained value problems for a linear peridynamic Navier equation. J. Elast. **116**, 27–51 (2014). DOI 10.1007/s10659-013-9456-z

[28] Oterkus, E., Madenci, E.: Peridynamic analysis of fiber reinforced composite materials. J. Mech. Mater. Struct. **7**, 45–84 (2012)

[29] Seleson, P., Gunzburger, M., Parks, M.L.: Interface problems in nonlocal diffusion and sharp transitions between local and nonlocal domains. Comput. Methd. Appl. Mech. Eng. **266**, 185–204 (2013)

[30] Silling, S.: Reformulation of elasticity theory for discontinuities and long-range forces. J. Mech. Phys. Solids **48**, 175–209 (2000)

[31] Silling, S., Lehoucq, R.B.: Peridynamic theory of solid mechanics. Adv. Appl. Mech. **44**, 73–168 (2010)

[32] Silling, S.A.: Dynamic fracture modeling with a meshfree peridynamic code. In: K. Bathe (ed.) Computational Fluid and Solid Mechanics, pp. 641–644. Elsevier, Oxford, UK (2003)

[33] Silling, S.A., Bobaru, F.: Peridynamic 3D problems of nanofiber networks and carbon nanotube-reinforced composites. In: Materials and Design: Proc. Numiform, American Institute of Physics, pp. 1565–1570. Melville, NY (2004)

[34] Silling, S.A., Bobaru, F.: Peridynamic modeling of membranes and fibers. Int. J. Nonlinear Mech. **40**, 395–409 (2005)

[35] Tian, X., Du, Q.: Analysis and comparison of different approximations to nonlocal diffusion and linear peridynamic equations. SIAM J. Numer. Anal. **51**(6), 3458–3482 (2013)

[36] Weidmann, J.: Linear Operator in Hilberträumen, Teil II: Anwendungen. Teubner (2003)

[37] Zhou, K., Du, Q.: Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions. SIAM J. Numer. Anal. **48**, 1759–1780 (2010)