RESULTS ON NONLOCAL BOUNDARY VALUE PROBLEMS *

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Abstract. In this article, we provide a variational theory for nonlocal problems where nonlocality arises due to interaction in a given horizon. With this theory, we prove well-posedness results for the weak formulation of nonlocal boundary value problems with Dirichlet, Neumann, and mixed boundary conditions for a class of kernel functions. The motivating application for nonlocal boundary value problems is the scalar stationary peridynamics equation of motion. The well-posedness results support practical kernel functions used in the peridynamics setting.

We also prove a spectral equivalence estimate which leads to a mesh size independent upper bound for the condition number of an underlying discretized operator. This is a fundamental conditioning result that would guide preconditioner construction for nonlocal problems. The estimate is a consequence of a nonlocal Poincaré-type inequality that reveals a horizon size quantification. We provide an example that establishes the sharpness of the upper bound in the spectral equivalence.

Keywords: Nonlocal operators, nonlocal boundary value problems, well-posedness, nonlocal Poincaré inequality, peridynamics, condition number, preconditioning.

1. Introduction

Nonlocal problems have become a critical part of modeling and simulation of complex phenomena that span vastly different length scales. Examples include evolution equations for species population densities [8], image processing [14, 24], porous media flow [9, 10, 22], and turbulence [4]. The book by Eringen [12] contains abundant applications which include fracture of solids, stress fields at dislocation cores and crack tips, and singularities present at concentrated loads (forces, couples, heat, etc.). Consequently, nonlocal models have become increasingly useful for multiscale modeling.

Our interest in nonlocal problems has been motivated by peridynamics which is *a nonlocal reformulation of continuum mechanics* developed by Silling [21]. Peridynamics employs an integral operator as opposed to differential operators to formulate the equations of motion, by allowing an interaction between points that are separated by a finite distance. Since there are *no assumptions made on the regularity of displacement or force fields*, the theory is suitable to study phenomena with discontinuities such fracture and fragmentation; see [21, 20]. The simplified governing equation

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in peridynamics theory has the form [21, eqn. (56)]:

$$\mathbf{u}_{tt}(\mathbf{x},t) = \mathcal{L}(\mathbf{u}(\mathbf{x},t)) - \mathbf{b}(\mathbf{x}), \tag{1.1}$$

where the linear nonlocal operator $\mathcal{L}$ is defined by:

$$\mathcal{L}(\mathbf{u})(\mathbf{x}) := -\int_{\Omega\cup\mathcal{B}\Omega} \mathbf{C}(\mathbf{x},\mathbf{x}')\left(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\right) d\mathbf{x}'. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with a nonlocal boundary $\mathcal{B}\Omega$ defined to be a fixed subset of the complement of Ω in $\mathbb{R}^d$. The measurable vector valued function $\mathbf{u}$ may represent a displacement field and *is not assumed to have any a priori regularity.* The kernel function $\mathbf{C}$ encodes physical material properties. Nonlocality arises due to the fact that points $\mathbf{x}' \neq \mathbf{x}$ can interact with $\mathbf{x}$. This leads us to define the notion of *horizon* of $\mathbf{x}$, $\mathcal{H}_{\mathbf{x}}$, as:

$$\mathcal{H}_{\mathbf{x}} := \{\mathbf{x}' \ : \ \|\mathbf{x}' - \mathbf{x}\| \leq \delta\}, \tag{1.3}$$

where $\delta > 0$ denotes the *horizon size.* We note that the *nonlocal boundary is a d-dimensional region* unlike its $(d-1)$-dimensional counterpart in local problems. For our purpose we take $\mathcal{B}\Omega$ to be the region surrounding Ω with unit thickness. That is, $\mathcal{B}\Omega := \{x \in \mathbb{R}^d \setminus \Omega : \text{dist}(x, \partial\Omega) \leq 1\ \}$, as depicted in Figure 1.1. We define the *nonlocal closure* of Ω by $\overline{\overline{\Omega}} := \Omega \cup \mathcal{B}\Omega$.

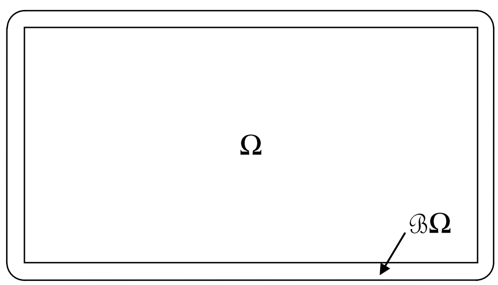

FIGURE 1.1. Typical domain for (1.4). $\mathbf{u}$ is prescribed in $\mathcal{B}\Omega$, and we solve for $\mathbf{u}$ in Ω.

In this article, for certain kernel functions C, we study scalar stationary nonlocal problems. An instance of such problems is the *scalar peridynamics equilibrium boundary value problem* whose strong formulation related to the nonlocal operator in (1.2) is given by:

$$\mathcal{L}(u)(\mathbf{x}) = b(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1.4}$$

where $u(\mathbf{x})$ is prescribed for $\mathbf{x} \in \mathcal{B}\Omega$ and b is a given function defined on $\overline{\overline{\Omega}}$. We aim to study the nonlocal problem (1.4) in a variational framework.

The first goal of this article is to prove *well-posedness* of the underlying variational form of (1.4). Utilizing a function space $V \subset L^2(\overline{\overline{\Omega}})$ that reflects the prescribed boundary condition, the variational formulation of (1.4) is the following: Given $b(\mathbf{x}) \in L^2(\overline{\overline{\Omega}})$, find $u(\mathbf{x}) \in V$ such that

$$a(u,v) = (b,v) \qquad \forall v \in V, \tag{1.5}$$

where

$$a(u,v) := -\int_{\overline{\overline{\Omega}}} \left\{ \int_{\overline{\overline{\Omega}}} C(\mathbf{x},\mathbf{x}')\,[u(\mathbf{x}') - u(\mathbf{x})]\; d\mathbf{x}' \right\} v(\mathbf{x})\; d\mathbf{x}'\; d\mathbf{x} \tag{1.6}$$

and $(\cdot,\cdot)$ is the standard L^2 inner product. We provide a collection of well-posedness results. In particular, we prove that (1.5) has a unique solution for purely Dirichlet, purely Neumann, and mixed nonlocal boundary conditions with continuous dependence upon the data, $b(x)$. For the pure Dirichlet problem, we provide details to the already known well-posedness result given in [15]. We utilize the connection provided by Brézis [7, p. 704] to establish the well-posedness of problems with purely Neumann and mixed boundary conditions.

The second goal is to *complement the conditioning analysis* given in the companion article [1]. In [1], the first author constructed a nonlocal domain decomposition framework to enable the usage of *scalable solvers*, in particular, iterative substructuring [6, 11, 13, 17, 16, 19], for the numerical solution of the boundary value problem (1.5). Condition number plays a crucial role in determining the effectiveness of an iterative method. Therefore, it is paramount to quantify the conditioning of the underlying discretized system by parameters such as the *horizon size*. We investigate the impact of the horizon size δ on the conditioning of the underlying operators. For that, we set C to be the *canonical* kernel function, $C(\mathbf{x},\mathbf{x}') = \chi_\delta(|\mathbf{x}-\mathbf{x}'|)$, whose only role is the representation of the horizon in (1.3) by a characteristic function. Namely,

$$\chi_\delta(|\mathbf{x}-\mathbf{x}'|) := \begin{cases} 1, & |\mathbf{x}-\mathbf{x}'| \le \delta \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

We prove the following spectral equivalence (with sufficiently small δ for the lower bound) for suitable function spaces V reflecting the underlying boundary conditions:

$$\underline{\lambda}(\overline{\overline{\Omega}})\,\delta^{d+2}\|u\|^2_{L^2(\overline{\overline{\Omega}})} \le a(u,u) \le \overline{\lambda}\,\delta^d\|u\|^2_{L^2(\overline{\overline{\Omega}})}, \quad u \in V. \tag{1.8}$$

The δ-quantification for both the lower and upper bounds in (1.8) is vital for the characterization of the condition number in terms of δ. The spectral equivalence (1.8) leads to the *remarkable result* that the condition number of the underlying stiffness matrix K can be bounded independently from the mesh size:

$$\kappa(K) \lesssim \delta^{-2}.$$

The main ingredient leading to the the δ-quantification is a *nonlocal characterization of Sobolev spaces* introduced by Bourgain, Brézis, and Mironescu [5]. We also utilize related results by Ponce [18] especially for the δ-quantification of the lower bound in (1.8). This nonlocal characterization of Sobolev spaces has been used for solving variational problems arising in image diffusion applications; see [3] and the references therein.

The remainder of the article is structured as follows. In Section 2, we construct the variational framework and prove the well-posedness of the boundary value problem with pure Dirichlet boundary value for general kernel functions that are practical in the peridynamics setting. In Section 3, we present the well-posedness proof for mixed and pure Neumann boundary conditions. In section 4, we give the δ-quantification for the spectral equivalence and establish the sharpness of the upper bound of (1.8) in 1D. In Section 5, we provide conclusions.

2. Variational problem

2.1. **Statement of the problem.** We assume that the kernel function is radial, i.e., $C(\mathbf{x}, \mathbf{x}') = \mathcal{C}(|\mathbf{x} - \mathbf{x}'|)$, where $\mathcal{C}(r)$ is a nonnegative and locally integrable function. In addition, we require

$$\mathcal{C}(r) > 0 \text{ on } (0, \delta), \tag{2.1}$$

for some $\delta > 0$. The condition (2.1) will be used in establishing the coercivity of the bilinear form for the Dirichlet case; see Lemma 2.4.

By abusing notation, we denote all forms of the kernel function $(C(\mathbf{x}, \mathbf{x}'), \mathcal{C}(|\mathbf{x}-\mathbf{x}'|), \mathcal{C}(r))$ simply by C. Observe that the following identity trivially holds

$$C(|\mathbf{x} - \mathbf{x}'|)\,[u(\mathbf{x}') - u(\mathbf{x})] = -C(|\mathbf{x}' - \mathbf{x}|)\,[u(\mathbf{x}) - u(\mathbf{x}')]. \tag{2.2}$$

for any scalar function u defined on $\overline{\overline{\Omega}}$. Physically equation (2.2) corresponds to Newton's third law. Using the antisymmetric property (2.2) and iterating the double integral, the bilinear form $a(u, v)$ in (1.6) can be equivalently written as follows:

$$a(u, v) = \frac{1}{2} \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} C(|\mathbf{x} - \mathbf{x}'|)\,[u(\mathbf{x}') - u(\mathbf{x})]\,[v(\mathbf{x}') - v(\mathbf{x})]\; d\mathbf{x}'\; d\mathbf{x}. \tag{2.3}$$

We will use the above form of the bilinear form throughout this article. The linear nonlocal variational problem is defined as follows: Given $b \in L^2(\overline{\overline{\Omega}})$, find $u \in V$ such that

$$a(u, v) = (b, v) \quad v \in V. \tag{2.4}$$

We will consider all types of boundary conditions and prove existence of a unique solution to (2.4) that depend continuously upon the data. The generic function space V is taken to be a subspace of $L^2(\overline{\overline{\Omega}})$ and is specified with respect to the type of boundary condition imposed. The function spaces of interest are as follows:

(1) *Pure Dirichlet boundary condition:*

$$V_D := \{v \in L^2(\overline{\overline{\Omega}}) : v = 0 \text{ on } \mathcal{B}\Omega\}.$$

Nonzero Dirichlet data can be reduced to this case following standard arguments.

(2) *Mixed boundary condition:*

$$V_M := \{v \in L^2(\overline{\overline{\Omega}}) : v = 0 \text{ on } \mathcal{B}\Omega_e\}.$$

The set $\mathcal{B}\Omega_e$ refers to the portion of the boundary where a Dirichlet boundary condition is imposed. We denote the complement $\mathcal{B}\Omega \setminus \mathcal{B}\Omega_e$ by $\mathcal{B}\Omega_n$. For this case we require that both $\mathcal{B}\Omega_e$ and $\mathcal{B}\Omega_n$ are open domains with positive measure volume.

(3) *Pure Neumann boundary condition:*

$$V_N := \{v \in L^2(\overline{\overline{\Omega}}) : \int_{\overline{\overline{\Omega}}} v\; d\mathbf{x} = 0\}.$$

We remark that to use standard results like the Riesz representation theorem, V needs to be a Hilbert space, which especially ensures the completeness property. This property is guaranteed due to the fact that V is a closed subspace of the Hilbert space $L^2(\overline{\overline{\Omega}})$. Indeed, V is closed for all types of boundary conditions. In the case of V_N, any strongly converging sequence also converges weakly. That means any converging sequence of functions with the same average will converge to a function with that average. In the cases of V_M and V_D, closedness follows from the fact that a strongly converging sequence has a subsequence that converges pointwise.

We also note that for the class of kernels considered in this article, (2.4) is the weak form of the Euler-Lagrange equation for the minimization problem:

$$\min_{u\in V} E(u), \tag{2.5}$$

where $E(u) := \frac{1}{2}a(u,u) - (b,u)$. A minimizer for problem (2.5) exists. Indeed, the function space V is weakly closed as it is a closed subspace of $L^2(\overline{\overline{\Omega}})$ and as it will be shown, $a(\cdot,\cdot)$ is a symmetric, coercive and bounded bilinear form on V. This implies that $a(u,u)$ defines a norm on V and that $E(u)$ is weakly lower semicontinuous. Then the existence of a minimizer follows from the direct method of calculus of variations, see of [23, Thm 1.2]. See also [15, 21] for related results.

2.2. **Well-posedness of the linear nonlocal variational problem: Dirichlet case.** Well-posedness of pure Dirichlet boundary value problems has been shown in [15]. However, the kernel functions are assumed to satisfy a sufficient condition that is too stringent. For applications of practical interest, this condition may lead to computational intractability, see [1]. As it is noted in [15] one can actually weaken the sufficient condition for a class of kernels. In this section will carry out the details of proving a well-posedness result based on these weekened sufficient conditions. Existence of a unique solution is also obtained in [25] for Dirichlet- and Neumann-type linear peridynamics boundary value problems.

Theorem 2.1. *For $a(\cdot,\cdot)$ given in (2.3) with C satisfying (2.1), the variational problem (2.4) with $V = V_D$ has a unique solution which satisfies the inequality*

$$\|u\|_{L^2} \le \Lambda \|b\|_{L^2},$$

for some constant $\Lambda = \Lambda(\delta) > 0$.

Proof. We observe that $a(\cdot,\cdot)$ is a symmetric bilinear form. Then the proof of the theorem follows from the Riesz representation theorem and the following two lemmas that prove boundedness and coercivity of $a(\cdot,\cdot)$ on the closed subspace V_D of the Hilbert space $L^2(\overline{\overline{\Omega}})$ □

Lemma 2.2. (Boundedness) *The bilinear form $a(\cdot,\cdot)$ given in (2.3) with C satisfying (2.1) is bounded on $L^2(\overline{\overline{\Omega}})$ with the estimate*

$$a(u,v) \le 2\,\overline{\beta}\,\|u\|_{L^2(\overline{\overline{\Omega}})}\|v\|_{L^2(\overline{\overline{\Omega}})},$$

where $\overline{\beta} := \sup_{x\in\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)d\mathbf{x}'$.

Proof. Let $u, v \in L^2(\overline{\overline{\Omega}})$. Then we obtain the following estimates:

$$a(u,v) = \frac{1}{2}\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)^{1/2}(u(\mathbf{x})-u(\mathbf{x}'))\, C(|\mathbf{x}-\mathbf{x}'|)^{1/2}(v(\mathbf{x})-v(\mathbf{x}'))\; d\mathbf{x}'\, d\mathbf{x}$$
$$\leq \frac{1}{2}\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(u(\mathbf{x})-u(\mathbf{x}'))^2\; d\mathbf{x}'\, d\mathbf{x}\right\}^{1/2}\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(v(\mathbf{x})-v(\mathbf{x}'))^2\; d\mathbf{x}'\, d\mathbf{x}\right\}^{1/2}$$
$$\leq \left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(u(\mathbf{x})^2+u(\mathbf{x}')^2)\; d\mathbf{x}'\, d\mathbf{x}\right\}^{1/2}\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(v(\mathbf{x})^2+v(\mathbf{x}')^2)\; d\mathbf{x}'\, d\mathbf{x}\right\}^{1/2}$$

Now we note that by a change in the order of integration and the fact that $C(|\mathbf{x}-\mathbf{x}'|)$ is an even (radial) function, one gets:

$$\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)u^2(\mathbf{x})\; d\mathbf{x}'\, d\mathbf{x} = \int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)u^2(\mathbf{x}')\; d\mathbf{x}'\, d\mathbf{x} \tag{2.6}$$

Then using (2.6), we have:

$$\begin{aligned} a(u,v) &\leq 2\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)u(\mathbf{x})^2\; d\mathbf{x}'\, d\mathbf{x}\right\}^{1/2}\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)v(\mathbf{x})^2\; d\mathbf{x}'\, d\mathbf{x}\right\}^{1/2} \\ &\leq 2\,\overline{\beta}\,\|u\|_{L^2(\overline{\overline{\Omega}})}\|v\|_{L^2(\overline{\overline{\Omega}})}, \end{aligned}$$

and $\overline{\beta} < \infty$ because C is a locally integrable function. ☐

Remark 2.3. We can fully quantify $\overline{\beta}$ in terms of δ for the canonical kernel $C(|\mathbf{x}-\mathbf{x}'|) = \chi_\delta(|\mathbf{x}-\mathbf{x}'|)$. Namely, $\overline{\beta} = v_d\delta^d$, v_d is the volume of the unit sphere in $\mathbb{R}^d$. This quantification leads to an explicit condition number upper bound in terms of δ; see the companion paper [1, Lemma 1]. The δ-quantification of $\overline{\beta}$ for general kernel functions will be also be provided in Corollary 4.1.

Lemma 2.4. (Coercivity) *Assume all the hypotheses of Theorem 2.1. Then, there exists* $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}}, \delta, C) > 0$ *such that*

$$\underline{\lambda}\,\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u) \quad \forall u \in V_D. \tag{2.7}$$

Proof. The proof is an extension of the one given in [2, Prop. 2.5] for a similar bilinear form. We construct a finite covering for $\overline{\overline{\Omega}}$ using strips of width $\frac{\delta}{2}$ as follows.

$$S_{-1} := \{\mathbf{x} \in \mathcal{B}\Omega : \mathrm{dist}(\mathbf{x}, \partial(\overline{\overline{\Omega}}) \leq \frac{\delta}{2}\}, \tag{2.8}$$

$$S_0 := \{\mathbf{x} \in \mathcal{B}\Omega \setminus S_{-1} : \mathrm{dist}(\mathbf{x}, S_{-1}) \leq \frac{\delta}{2}\}, \tag{2.9}$$

$$S_1 := \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \partial\Omega) \leq \frac{\delta}{2}\}, \tag{2.10}$$

$$S_j := \{\mathbf{x} \in \Omega \setminus \bigcup_{k=1}^{j-1} S_k : \mathrm{dist}(\mathbf{x}, S_{j-1}) \leq \frac{\delta}{2}\}, \quad j = 1\ldots, l. \tag{2.11}$$

Note that the number of strips is $l = l(\overline{\overline{\Omega}}, \delta)$. We trivially have the following for $j = 0, \dots, l$:

$$\int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} C(|\mathbf{x} - \mathbf{x}'|) \, |u(\mathbf{x}') - u(\mathbf{x})|^2 \, d\mathbf{x}' d\mathbf{x} \geq \int_{S_j} \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \, |u(\mathbf{x}') - u(\mathbf{x})|^2 \, d\mathbf{x}' d\mathbf{x}.$$

Note that since C is radial and locally integrable,

$$\int_{S_j} C(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' \leq \|C\|_{L^1(B(\mathbf{0},R))}, \quad \forall \; \mathbf{x} \in \overline{\overline{\Omega}}, \tag{2.12}$$

where R is taken to be the diameter of $\overline{\overline{\Omega}}$ and $B(\mathbf{0}, R)$ is the ball of radius R centered at the origin. A change in the order of integration, the inequality $|u(\mathbf{x})|^2 \leq 2\{|u(\mathbf{x}') - u(\mathbf{x})|^2 + |u(\mathbf{x}')|^2\}$, and (2.12) yield the following:

$$\begin{aligned}
& \int_{S_j} \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \, |u(\mathbf{x}') - u(\mathbf{x})|^2 \; d\mathbf{x}' d\mathbf{x} \\
\geq \; & \frac{1}{2} \int_{S_j} \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \, |u(\mathbf{x})|^2 \; d\mathbf{x}' d\mathbf{x} - \int_{S_j} \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \, |u(\mathbf{x}')|^2 \; d\mathbf{x}' d\mathbf{x} \\
= \; & \frac{1}{2} \int_{S_j} \left\{ \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' \right\} |u(\mathbf{x})|^2 \; d\mathbf{x} - \int_{S_{j-1}} \left\{ \int_{S_j} C(|\mathbf{x} - \mathbf{x}'|) d\mathbf{x} \right\} |u(\mathbf{x}')|^2 \; d\mathbf{x}' \\
\geq \; & \frac{1}{2} \int_{S_j} \left\{ \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' \right\} |u(\mathbf{x})|^2 \; d\mathbf{x} - \|C\|_{L^1(B(\mathbf{0},R))} \int_{S_{j-1}} |u(\mathbf{x}')|^2 \; d\mathbf{x}' \\
\geq \; & \frac{1}{2} \min_{x \in \overline{S}_j} \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' \int_{S_j} |u(\mathbf{x})|^2 \; d\mathbf{x} - \|C\|_{L^1(B(\mathbf{0},R))} \int_{S_{j-1}} |u(\mathbf{x}')|^2 \; d\mathbf{x}'.
\end{aligned}$$

The function

$$F(\mathbf{x}) := \int_{S_{j-1}} C(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}', \quad \mathbf{x} \in \overline{S_j}$$

is continuous due to the fact that $C(|\mathbf{x}-\mathbf{x}'|)$ is integrable and the continuity of the integral operator. By construction of the covering, we have $S_{j-1} \cap B(\mathbf{x}, \delta) \neq \emptyset, \quad \mathbf{x} \in S_j$, and hence, we obtain

$$F(\mathbf{x}) > 0, \quad \mathbf{x} \in \overline{S_j},$$

since $B(\mathbf{x}, \delta)$ is in support of $C(\mathbf{x} - \mathbf{x}')$ by (2.1). Thus, $F(\mathbf{x})$ attains its infimum in $\overline{S_j}$. We denote the minimum value by

$$\alpha_j := \min_{\mathbf{x} \in \overline{S}_j} F(\mathbf{x}) > 0.$$

Consequently, we have the following inequality:

$$\frac{\alpha_j}{2} \int_{S_j} |u(\mathbf{x})|^2 \; d\mathbf{x} \leq a(u, u) + \|C\|_{L^1(B(\mathbf{0},R))} \int_{S_{j-1}} |u(\mathbf{x}')|^2 \; d\mathbf{x}'. \tag{2.13}$$

Since $u \in V_D$, $u = 0$ on $\mathcal{B}\Omega$,

$$\int_{S_{-1}} |u(\mathbf{x})|^2 \; d\mathbf{x} = \int_{S_0} |u(\mathbf{x})|^2 \; d\mathbf{x} = 0.$$

Applying the above to (2.13) we get

$$\frac{\alpha_1}{2}\int_{S_1} |u(\mathbf{x})|^2 \, d\mathbf{x} \leq 2a(u,u) \tag{2.14}$$

For the cases $j = 2, 3$, we respectively have:

$$\frac{\alpha_2}{2}\int_{S_2} |u(\mathbf{x})|^2 \, d\mathbf{x} \quad\leq\quad 2a(u,u) + \|C\|_{L^1(B(\mathbf{0},R))} \int_{S_1} |u(\mathbf{x}')|^2 \, d\mathbf{x}' \tag{2.15}$$

$$\frac{\alpha_3}{2}\int_{S_3} |u(\mathbf{x})|^2 \, d\mathbf{x} \quad\leq\quad 2a(u,u) + \|C\|_{L^1(B(\mathbf{0},R))} \int_{S_2} |u(\mathbf{x}')|^2 \, d\mathbf{x}'. \tag{2.16}$$

Substituting (2.14) into (2.15) and then (2.15) into (2.16) we obtain

$$\frac{\alpha_3}{2}\int_{S_3} |u(\mathbf{x})|^2 \, d\mathbf{x} \leq 2\Big(1 + \frac{2\,\|C\|_{L^1(B(\mathbf{0},R))}}{\alpha_2} + \frac{2\,(\|C\|_{L^1(B(\mathbf{0},R))})^2}{\alpha_2\,\alpha_1}\Big)\; a(u,u).$$

Continuing this process we see the existence of a constant $c(\overline{\overline{\Omega}}, \delta, C)$ satisfying:

$$\frac{\alpha_j}{2}\int_{S_j} |u(\mathbf{x})|^2 \, d\mathbf{x} \leq c(\overline{\overline{\Omega}}, \delta, C)\; a(u,u), \quad j = -1, \ldots, l. \tag{2.17}$$

Adding (2.17) for $j = -1, \ldots, l$ and using the fact that the covering of $\overline{\overline{\Omega}}$ is composed of disjoint strips, i.e., $\overline{\overline{\Omega}} = \cup_{k=-1}^{l} S_k,\; S_j \cap S_k = \emptyset,\; j \neq k$, we arrive at the coercivity result. □

Remark 2.5. The above proof can be extended, as is done in [2, Prop. 2.5], to get the estimate

$$\underline{\lambda}(\overline{\overline{\Omega}}, \delta, C)\, \|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u) + \int_{S_{-1}} |u(\mathbf{x})|^2 \, d\mathbf{x}, \tag{2.18}$$

for functions $u \in L^2(\overline{\overline{\Omega}})$ that do not necessarily vanish on $\mathcal{B}\Omega$, where S_{-1} is the outermost strip of the covering of $\overline{\overline{\Omega}}$. However deducing a coercivity estimate from (2.18) does not seem possible unless we assume a zero Dirichlet condition on the nonlocal boundary. Thus we may not apply (2.18) for establishing coercivity of the mixed and Neumann problems.

We also notice that, from the Riesz representation theorem, the constant Λ in Theorem 2.1 is equal to $1/\underline{\lambda}(\overline{\overline{\Omega}}, \delta)$, with an obvious dependence on the horizon size δ. The explicit quantification of this dependence of Λ on δ will be proved in the next section.

3. Well-posedness for Mixed and Pure Neumann problems

In this section we will obtain a coercivity estimate for the mixed and Neumann problems for a class of kernel functions. We will also set the mathematical background for the spectral equivalence estimate.

3.1. **Review of nonlocal Poincaré-type inequality.** We begin by reviewing some estimates obtained in [5] and [18] on the nonlocal characterization of Sobolev functions. Before stating these nonlocal Poincaré -type inequalities we recall the standard local Poincaré's inequality:

$$\|u\|_{L^2(\overline{\overline{\Omega}})} \leq c_{Pncr} \|\nabla u\|_{L^2(\overline{\overline{\Omega}})} \tag{3.1}$$

holds true for all $u \in H^1(\overline{\overline{\Omega}})$ satisfying either

$$\int_{\overline{\overline{\Omega}}} u \; d\mathbf{x} = 0 \quad \text{or} \quad |\{\mathbf{x} \in \overline{\overline{\Omega}} : u(\mathbf{x}) = 0\}| = \mu > 0.$$

The constant c_{Pncr} depends only on d, μ and $\overline{\overline{\Omega}}$ and we always assume that it is the best constant. The nonlocal Poincaré-type inequality obtained in [5] and later improved in [18] utilizes the sequence of radial functions ρ_n satisfying the following conditions:

$$\rho_n \geq 0 \text{ a.e. in } \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \rho_n = 1, \quad \forall n \geq 1, \quad \text{and} \lim_{n\to\infty} \int_{|h|>r} \rho_n(h) dh = 0, \quad \forall r > 0. \tag{3.2}$$

The following is the nonlocal Poincaré-type inequality proved in [18].

Lemma 3.1. *For any $\eta > 0$, there exists n_0 such that*

$$\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq (\frac{c_{Pncr}}{k_d} + \eta) \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}) - u(\mathbf{x}')|^2}{|\mathbf{x} - \mathbf{x}'|^2} \rho_n(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' d\mathbf{x} \tag{3.3}$$

for all $u \in V_N$ and $n \geq n_0$. Here k_d is a constant that depends only on d.

Note that inequality (3.3) in Lemma 3.1 holds true for functions in $L^2(\overline{\overline{\Omega}})$ with zero average. We extend Lemma 3.1 to functions that vanish on a nontrivial subset of $\overline{\overline{\Omega}}$ in the following. [1]

Lemma 3.2. *For any $\eta > 0$, there exists n_0 such that*

$$\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq (\frac{c_{Pncr}}{k_d} + \eta) \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}) - u(\mathbf{x}')|^2}{|\mathbf{x} - \mathbf{x}'|^2} \rho_n(|\mathbf{x} - \mathbf{x}'|) \; d\mathbf{x}' d\mathbf{x},$$

for all $u \in V_M$ and $n \geq n_0$. Here, k_d is a constant that depends only on d.

We remark that in the above two lemmas the choice of n_0 not only depends on η but also on Ω, the sequence of radial functions ρ_n used and the subspaces V_N and V_M. We also note that the nonlocal Poincaré-type estimates hold for functions that belong to closed subspaces of $L^2(\overline{\overline{\Omega}})$. However, the standard local Poincaré's inequality applies to closed subspaces of $H^1(\overline{\overline{\Omega}})$, which requires more regularity than $L^2(\overline{\overline{\Omega}})$.

The proof of Lemma 3.2 uses a similar line of argument as in that of Lemma 3.1; see [18]. We begin with the following extension of a compactness result.

[1] Recall that $\mathcal{B}\Omega_e$ is a subset of $\mathcal{B}\Omega$ whose volume measure is positive.

Lemma 3.3. *If $(u_n) \subset V_M$ is a bounded sequence in $L^2(\overline{\overline{\Omega}})$ such that*

$$\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} \frac{|u_n(\mathbf{x}) - u_n(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^2}\rho_n(|\mathbf{x}-\mathbf{x}'|)\ d\mathbf{x}'d\mathbf{x} \le b_0 \quad \forall n \ge 1,$$

then the following statements hold:

(i) *(u_n) is relatively compact in $L^2(\overline{\overline{\Omega}})$.*
(ii) *If $u \in L^2(\overline{\overline{\Omega}})$ and $u_{n_j} \to u$ in $L^2(\overline{\overline{\Omega}})$, then $u \in H^1(\overline{\overline{\Omega}}) \cap V_M$.*
(iii) *Moreover, the limit function u satisfies the following gradient estimate: $\int_{\overline{\overline{\Omega}}} |\nabla u|^2 \le b_0/k_d$.*

Proof. The results follow from [18, Thm 1.2] and the fact that V_M is a closed subspace of $L^2(\overline{\overline{\Omega}})$. □

Proof of Lemma 3.2. We argue by contradiction. Assume that the lemma is false. Then there exists $c_0 > c_{Pncr}/k_d$, and a sequence $u_n \in V_M$ with the property that

$$\|u_n\|^2_{L^2(\overline{\overline{\Omega}})} = 1 \quad \text{and} \quad \int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} \frac{|u_n(\mathbf{x}) - u_n(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^2}\rho_n(|\mathbf{x}-\mathbf{x}'|)\ d\mathbf{x}'\,d\mathbf{x} \le 1/c_0.$$

Observe that (u_n) satisfies the assumption of Lemma 3.3 and so we can extract a subsequence that converges strongly in L^2 to a function $u \in H^1 \cap V_M$. Moreover,

$$\|u\|^2_{L^2(\overline{\overline{\Omega}})} = 1, \quad \text{and} \quad \int_{\overline{\overline{\Omega}}} |\nabla u|^2\ d\mathbf{x} \le 1/(c_0 k_d) < 1/c_{Pncr},$$

since $c_0 > c_{Pncr}/k_d$. But this contradicts (3.1). □

3.2. **The Mixed and Pure Neumann variational problems are well-posed.** We are now ready to state the well-posedness of the pure Neumann and mixed problems for a restricted class of kernels. In addition to the requirements of local integrability and (2.1), we also assume that the kernel is of the form:

$$C(\mathbf{x},\mathbf{x}') = \gamma(\frac{|\mathbf{x}-\mathbf{x}'|}{\delta}) \tag{3.4}$$

where

$$\gamma \ge 0, \quad \text{supp}(\gamma) \subset (0,2), \quad \gamma(r)r^{d-1} \in L^1_{\text{loc}}(0,+\infty), \quad \text{and} \int_0^\infty \gamma(r)r^{d+1}\ dr = 1. \tag{3.5}$$

Then, a simple calculation yields that the sequence $\rho_\delta(r)$ defined by

$$\rho_\delta(r) := \frac{1}{\omega_d \delta^{d+2}}\gamma(\frac{r}{\delta})r^2$$

satisfies (3.2) and

$$\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}) - u(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^2}\rho_\delta(|\mathbf{x}-\mathbf{x}'|)\ d\mathbf{x}'d\mathbf{x} = \frac{1}{\omega_d\delta^{d+2}}a(u,u).$$

In the above ω_d is the surface area of the unit sphere in $\mathbb{R}^d$.

Now application of Lemma 3.1 and Lemma 3.2 for this sequence establishes the coercivity of the bilinear form $a(\cdot,\cdot)$ on V_M and V_N.

Corollary 3.4. (Coercivity) *For $a(\cdot,\cdot)$ given in (2.3) and C satisfying (2.1) and (3.4), there exist $\delta_0 = \delta_0(\overline{\overline{\Omega}},\gamma) > 0$ and $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}},\delta_0)$ such that for all $0<\delta<\delta_0$ and $u \in V_M$ or V_N:*

$$\underline{\lambda}\,\delta^{d+2}\,\|u\|^2_{L^2(\overline{\overline{\Omega}})} \le a(u,u).$$

Now the well-posedness of the mixed or Neumann variational problem follows from the Riesz representation theorem.

Theorem 3.5 (Well-posedness)**.** *For $a(\cdot,\cdot)$ given in (2.3) and C satisfying (2.1) and (3.4) the variational problem (2.4) with $V = V_N$ or $V = V_M$ has a unique solution which satisfies the inequality*

$$\|u\|_{L^2} \le \Lambda \|b\|_{L^2},$$

for some constant $\Lambda > 0$.

Remark 3.6 (Example). A constant multiple of the truncated power function

$$\gamma(r) = r^\alpha\,\chi_{[0,2]}(r)$$

satisfies (3.5) if and only if $\alpha > -d$. Other examples of radial functions of the form,

$$\gamma(r) = \ln(r)\,r^\alpha\,\chi_{[0,2]}(r),$$

can also be constructed.

4. Spectral equivalence and sharpness of the upper bound

Combining Lemma 2.2 and Lemma 3.4, we arrive at the main spectral equivalence result used in the companion article [1].

Corollary 4.1. *For $a(\cdot,\cdot)$ given in (2.3) and C satisfying (2.1) and (3.4) there exist $\delta_0 > 0$, $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}},\delta_0)$ and $\overline{\lambda} = \overline{\lambda}(\gamma,d)$ such that for all $0<\delta<\delta_0$ and $u \in V_D, V_M$, or V_N, we have*

$$\underline{\lambda}\,\delta^{d+2}\,\|u\|^2_{L^2(\overline{\overline{\Omega}})} \le a(u,u) \le \overline{\lambda}\delta^d \|u\|^2_{L^2(\overline{\overline{\Omega}})}. \tag{4.1}$$

Proof. The only part that needs proof is the upper bound and it follows from δ-quantification of the upper bound β found in Lemma 2.2. To that end, let R be the diameter of $\overline{\overline{\Omega}}$. Then for any $\mathbf{x} \in \overline{\overline{\Omega}}$, we have $\overline{\overline{\Omega}} \subset B(\mathbf{x},R)$ and

$$\beta \le \sup_{x\in\overline{\overline{\Omega}}} \int_{B(\mathbf{x},R)} C(|\mathbf{x}-\mathbf{x}'|)d\mathbf{x}' = \int_{B(\mathbf{0},R)} C(|\mathbf{x}'|)d\mathbf{x}'$$

Since C is radial, we can estimate the last integral as

$$\int_{B(\mathbf{0},R)} C(|\mathbf{x}'|)d\mathbf{x}' = \int_{B(\mathbf{0},R)} \gamma(\frac{|\mathbf{x}'|}{\delta})d\mathbf{x}' = \omega_d\delta^d \int_0^{R/\delta} \gamma(s)s^{d-1}ds \le \overline{\lambda}\delta^d.$$

In the above we have integrated in polar coordinates and ω_d is the surface area of the unit sphere in $\mathbb{R}^d$. Moreover, the constant $\overline{\lambda}$ depends on the diameter of $\overline{\overline{\Omega}}$ and the L^1-norm of $\gamma(r)r^{d-1}$. $\square$

Remark 4.2. The spectral equivalence (4.1) is the result needed in the companion article [1] and it leads to a remarkable conditioning result. Namely, the condition number of the discretized operator can be bounded independently from the mesh size; $\kappa(K) \lesssim \delta^{-2}$, where K is a stiffness matrix. This is a pioneering fundamental result which would guide preconditioner construction for nonlocal problems.

We complete this section by providing a 1D example that establishes the sharpness of the δ-quantification in the upper bound by using the following piecewise constant function $\overline{\overline{\Omega}} = [-1, 2]$:

$$u(x) := \begin{cases} 1, & x \in [0, \delta] \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

We utilize the canonical kernel $C(x, x') = \chi_\delta(|x - x'|)$ as in (1.7). Note that $a(u, u)$ is identically zero in the domain of integration

$$(x, x') \in [-1, 2] \times [x - \delta, x + \delta] \cap [-1, 2], \tag{4.3}$$

except the following regions:

$$\begin{array}{lccccllll}
\text{Region I:} & (x,x') & \in & [-\delta, 0] & \times & [0, x+\delta]; & u(x) = 0, & u(x') = 1; & a(u,u) = \delta^2/4 \\
\text{Region II:} & (x,x') & \in & [0, \delta] & \times & [x-\delta, 0]; & u(x) = 1, & u(x') = 0; & a(u,u) = \delta^2/4 \\
\text{Region III:} & (x,x') & \in & [0, \delta] & \times & [\delta, x+\delta]; & u(x) = 0, & u(x') = 1; & a(u,u) = \delta^2/4 \\
\text{Region IV:} & (x,x') & \in & [\delta, 2\delta] & \times & [x-\delta, \delta]; & u(x) = 0, & u(x') = 1; & a(u,u) = \delta^2/4 \\
\text{Region IV:} & (x,x') & \in & [0, \delta] & \times & [0, \delta]; & u(x) = 1, & u(x') = 1; & a(u,u) = 0.
\end{array}$$

Adding contributions from all regions, we conclude that

$$a(u, u) = \delta^2.$$

On the other hand,

$$\|u\|^2_{L^2(\overline{\overline{\Omega}})} = \int_0^\delta dx = \delta.$$

Finally, we see that the Rayleigh quotient becomes

$$\frac{a(u, u)}{\|u\|^2_{L^2(\overline{\overline{\Omega}})}} = \delta, \tag{4.4}$$

implying that the upper bound in Corollary 4.1 holds with $\overline{\lambda} = 1$.

In the companion article [1], we also establish the sharpness of the lower bound numerically by using piecewise constant and piecewise linear finite element discretizations.

5. Conclusion

We provided a variational theory for nonlocal problems and proved the well-posedness of the weak formulation of nonlocal boundary value problems with Dirichlet, Neumann, and mixed boundary conditions for general kernel functions that are practical in the peridynamics setting.

We proved a nonlocal Poincaré inequality which reveals the horizon size quantification by utilizing a nonlocal characterization of Sobolev spaces. Then, we used this quantification to prove a spectral equivalence estimate which leads to a *mesh size independent upper bound for the condition number* of the underlying discretized operator. The quantification is the complementing result needed in the companion article [1]. Hence, we proved *the first fundamental conditioning result* that would guide preconditioner construction for nonlocal problems. We established the sharpness of the upper bound in the spectral equivalence by providing an example.

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