Variational Theory and Domain Decomposition for Nonlocal Problems

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We provide a variational theory for nonlocal problems where nonlocality arises due to interaction in a given neighborhood. In [1], we prove well-posedness results for the weak formulation of nonlocal boundary value problems with Dirichlet, Neumann, and mixed boundary conditions for a class of kernel functions. We also prove the following spectral equivalence:

$$\underline{\lambda}\,\delta^{d+2}\,\|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u) \leq \overline{\lambda}\delta^{d}\|u\|^2_{L^2(\overline{\overline{\Omega}})}, \tag{1}$$

where δ denotes the size of the neighborhood and $a(\cdot,\cdot)$ denotes the underlying bilinear form. The spectral equivalence (1) leads to the *remarkable result* that the condition number of the underlying stiffness matrix $K_{nonlocal}$ can be bounded independently from the mesh size h:

$$\kappa(K_{nonlocal}) \lesssim \delta^{-2}.$$

This is a fundamental conditioning result that would guide preconditioner construction for nonlocal problems. The equivalence (1) is a consequence of a nonlocal Poincaré-type inequality that reveals neighborhood size quantification. We provide an example that establishes the sharpness of the upper bound in (1).

In [2], we introduce a nonlocal two-domain variational formulation utilizing nonlocal transmission conditions, and prove equivalence with the single-domain formulation. A nonlocal Schur complement $S_{nonlocal}$ is introduced. We comparatively study the conditioning of $K_{nonlocal}$ and the (local) discrete Laplace systems. In the local case, the condition number of the stiffness matrix and the corresponding Schur complement matrix vary with h^{-2} and h^{-1}, respectively. We provide numerical experiments demonstrating the conditioning of the nonlocal one- and two-domain problems. For a fixed h, $0 < h \ll \delta$, we demonstrate that $K_{nonlocal}$ and the corresponding $S_{nonlocal}$ vary with δ^{-2} and δ^{-1}, respectively. In order to validate the numerical results, we establish condition number bounds for $K_{nonlocal}$ and the corresponding $S_{nonlocal}$.

References

[1] B. Aksoylu and T. Mengesha, *Results on nonlocal boundary value problems*, Numerical Functional Analysis and Optimization, 31 (2010), pp. 1301–1317.

[2] B. Aksoylu and M. L. Parks, *Variational theory and domain decomposition for nonlocal problems*, Applied Mathematics and Computation, 217 (2011), pp. 6498–6515.