Chapter 15
Numerical study of the high-contrast Stokes equation and its robust preconditioning

Burak Aksoylu and Zuhal Unlu

Abstract We numerically study the Stokes equation with high-contrast viscosity coefficients. The high contrast viscosity values create complications in the convergence of the underlying solver methods. To address this complication, we construct a preconditioner that is robust with respect to contrast size and mesh size simultaneously based on the preconditioner proposed by Aksoylu et al. (2008, Comput. Vis. Sci. 11, pp. 319–331). We examine the performance of our preconditioner against multigrid and provide a comparative study reflecting the effect of the underlying discretization and the aspect ratio of the mesh by utilizing the preconditioned inexact Uzawa and Minres solvers. Our preconditioner turns out to be most effective when used as a preconditioner to the inexact p-Uzawa solver and we observe contrast size and mesh size robustness simultaneously. As the contrast size grows asymptotically, we numerically demonstrate that the inexact p-Uzawa solver converges to the exact one. We also observe that our preconditioner is contrast size and mesh size robust under p-Minres when the Schur complement solve is accurate enough. In this case, the multigrid preconditioner loses both contrast size and mesh size robustness.

15.1 Introduction

The Stokes equation plays a fundamental role in the modeling of several problems in emerging geodynamics applications. Numerical solutions to the Stokes flow prob-

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lems especially with high-contrast variations in viscosity is critically needed in the computational geodynamics community; see recent studies [26, 27, 34, 44]. The high-contrast viscosity corresponds to a small Reynolds number regime because the Reynolds number is inversely proportional to the viscosity value. One of the main applications of the high-contrast Stokes equation is the study of earth's mantle dynamics. The processes such as the long time scale dynamics of the earth's convecting mantle, the formation and subsequent evolution of plate tectonics can be satisfactorily modeled by the Stokes equation; see [26, 34, 35] for further details. Realistic simulation of mantle convection critically relies on the treatment of the two essential components of simulation: *the contrast size in viscosity* and *the mesh resolution*. Hence, our aim is to achieve robustness of the underlying preconditioner with respect to the contrast size and the mesh size simultaneously, which we call as *m*- and *h*-robustness, respectively.

Roughness of PDE coefficients causes loss of robustness of preconditioners. In [3, 4] Aksoylu and Beyer have studied the diffusion equation with such coefficients in the operator theory framework and have showed that the roughness of coefficients creates serious complications. For instance, in [4] they have shown that the standard elliptic regularity in the smooth coefficient case fails to hold. Moreover, the domain of the diffusion operator heavily depends on the regularity of the coefficients. Similar complications also arise in the Stokes case. This article came about from a need to address solver complications through the help of robust preconditioning. For that, we construct a robust preconditioner based on the one proposed in [2], which we call as the Aksoylu-Graham-Klie-Scheichl (AGKS) preconditioner. The AGKS preconditioner originates from the family of robust preconditioners constructed in [5]. It was proven and numerically verified to be *m*- and *h*-robust simultaneously.

The AGKS preconditioner was originally designed for the high-contrast diffusion equation under finite element discretization. In [6] we extended the AGKS preconditioner from finite element discretization to cell-centered finite volume discretization. Hence, we have shown that the same preconditioner could be used for different discretizations with minimal modification. Furthermore, in [7] we applied the same family of preconditioners to high-contrast biharmonic plate equation. Therefore, we have accomplished a desirable preconditioning design goal by using the same family of preconditioners to solve the elliptic family of PDEs with varying discretizations. In this article, we aim to bring the same preconditioning technology to *vector valued* problems such as the Stokes equation. We extend the usage of AGKS preconditioner to the solution of the stationary Stokes equation in a domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} -\nabla \cdot (\nu \nabla u) + \nabla p &= f \quad \text{in} \quad \Omega, \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega, \end{aligned} \tag{15.1}$$

with piecewise constant high-contrast viscosity used in the slab subduction referred as the *Sinker* model by [34]:

$$\nu(x) = \begin{cases} m \gg 1, & x \in \Omega_H, \\ 1, & x \in \Omega_L. \end{cases} \tag{15.2}$$

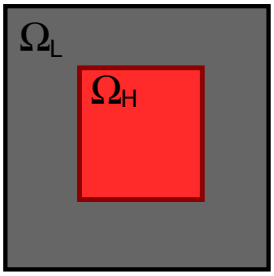

Fig. 15.1: $\Omega = \overline{\Omega}_H \cup \Omega_L$ where Ω_H and Ω_L are highly- and lowly-viscous regions, respectively.

Here, u, p, and f stand for the velocity, pressure, and body force, respectively.

The discretization of (15.1) gives rise to the following saddle point matrix:

$$\mathscr{A}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} K(m) & B^t \\ B & 0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{15.3}$$

The velocity vector can be treated componentwise which allows the usage of a single finite element space for each component. The extension of AGKS preconditioner from diffusion to Stokes equation is accomplished by the following crucial block partitioning of (15.3); see [20, p. 226]:

$$\begin{bmatrix} K^x(m) & 0 & (B^x)^t \\ 0 & K^y(m) & (B^y)^t \\ B^x & B^y & 0 \end{bmatrix}\begin{bmatrix} u^x \\ u^y \\ p \end{bmatrix} = \begin{bmatrix} f^x \\ f^y \\ 0 \end{bmatrix}, \tag{15.4}$$

where $K^* = K^x = K^y$ are the scalar diffusion matrices, and B^x and B^y represent the weak derivatives in x and y directions, respectively. We apply the AGKS preconditioning idea to the K^x and K^y blocks by further decomposing each of them as the following 2×2 block system; see [7, Eqn. 11], [6, Eqn. 4], [2, Eqn. 3]:

$$K^*(m) = \begin{bmatrix} K^*_{HH}(m) & K^*_{HL} \\ K^*_{LH} & K^*_{LL} \end{bmatrix}, \tag{15.5}$$

where the degrees of freedom (DOF) are identified as *high* and *low* based on the viscosity value in (15.2) and $K^*_{HH}, K^*_{HL}, K^*_{LH}$, and K^*_{LL} denote couplings between the high-high, high-low, low-high, and low-low DOF, respectively. The exact inverse of K^* can be written as:

$$K^{*^{-1}} = \begin{bmatrix} I_{HH} & -K^{*^{-1}}_{HH}K^*_{HL} \\ 0 & I_{LL} \end{bmatrix}\begin{bmatrix} K^{*^{-1}}_{HH} & 0 \\ 0 & S^{*^{-1}} \end{bmatrix}\begin{bmatrix} I_{HH} & 0 \\ -K^*_{LH}K^{*^{-1}}_{HH} & I_{LL} \end{bmatrix},$$

where I_{HH} and I_{LL} denote the identity matrices of the appropriate dimension and the Schur complement S^* is explicitly given by:

$$S^*(m) = K^*_{LL} - K^*_{LH} K^{*^{-1}}_{HH}(m) K^*_{HL}. \tag{15.6}$$

The AGKS preconditioner is defined as follows:

$$\hat{K}^{*^{-1}}(m) := \begin{bmatrix} I_{HH} & -K^{\infty^\dagger}_{HH} K^*_{HL} \\ 0 & I_{LL} \end{bmatrix} \begin{bmatrix} K_{HH}(m)^{*^{-1}} & 0 \\ 0 & S^{\infty^{-1}} \end{bmatrix} \begin{bmatrix} I_{HH} & 0 \\ -K^*_{LH} K^{\infty^\dagger}_{HH} & I_{LL} \end{bmatrix}, \tag{15.7}$$

where $K^{\infty^\dagger}_{HH}$ and S^∞ are the asymptotic values of $K^{*^{-1}}_{HH}$ and S^*, respectively; see [2, Lemma 1].

15.1.1 Literature Review

There are many solution methods proposed for the system of equations in (15.3); see the excellent survey article [13]. Based on where the emphasis is put in the design of a solution method, solving a saddle-point matrix system can be classified into two approaches: *preconditioning and solver*. The *preconditioning approach* aims to construct novel preconditioners for standard solver methods such as Uzawa and Minres. A vast majority of the articles on the *preconditioning* approach focuses on the preconditioning of Schur complement matrix; see [17, 31, 32, 34, 37, 39, 44]. It is well known that the Schur complement matrix S is spectrally equivalent to the pressure mass matrix (PMM) for the steady Stokes equation; see [16]. For rigorous convergence analysis of Krylov solvers with PMM preconditioner, see [42, 45]. Elman and Silvester [23] established that scaled PMM lead to h-robustness for the Stokes equation with large constant viscosity. Using a new inner product, Olshanskii and Reusken [37] introduced a robust preconditioner for the Schur complement matrix $S = BK^{-1}B^t$ for discontinuous viscosity $0 < \nu \leq 1$ and showed that the preconditioned Uzawa (p-Uzawa) and Minres (p-Minres) became h-robust with this new PMM preconditioner. Further properties of this preconditioner such as clustering in the spectrum of preconditioned S system was shown in [29]. It was pointed out in [31] that Elman [18] designed LSQR commutator (BFBt) preconditioner in order to overcome the m-robustness issues by using $\hat{S} = (BB^t)^{-1}BKB^t(BB^t)^{-1}$ preconditioner for S. This preconditioner is further studied in [17, 19]. Additionally, the usage of variants of the BFBt preconditioner for the high-contrast Stokes equation is popularized with $\nu|_{\Omega_H} = m \gg 1$ in geodynamics applications in [26, 27, 34, 44]. May and Moresi [34] established that this preconditioner was m-robust when used along with a preconditioned Schur Complement Reduction solver and h-robustness of this preconditioner when used with the Schur method and generalized conjugate residual method with block triangular preconditioners was obtained by a further study in [44].

There have been studies focusing on different ways of preconditioning K for the Stokes equation restricted to constant viscosity case; see [15, 22, 43]. It was observed that a single multigrid (MG) cycle with an appropriate smoother was usually a good preconditioner for K because MG is sufficiently effective as a preconditioner

for the constant viscosity case; see [20]. For discontinuous coefficient case, however, there has not been much study to analyze the performance of preconditioners for K in a Stokes solver framework. Since MG loses h-robustness, there is an imminent need for the robustness study of preconditioners for the case of discontinuous coefficients and we present the AGKS preconditioner to address this need.

The *solver method approach* aims to construct a solver by sticking with standard preconditioners such as MG for the K matrix and PMM or BFBt for the S matrix. The performance of the solver depends heavily on the choice of the inner preconditioner; see [9, 10, 22, 25]. The Uzawa solver is one of the most popular iterative methods for the saddle point problems in fluid dynamics; see [25, 28, 30]. Since this method requires the solution of K system in each step, this leads to the utilization of an inexact Uzawa method involving an approximate evaluation of K^{-1}; see [11, 46]. This method involves an inner and outer iteration (in our context, S- and outer-solve, respectively), and the convergence of this method is studied extensively in [11, 15, 22, 39].

Another commonly used iterative method is Minres; see [38]. The usage of block diagonal preconditioner for the p-Minres solver was suggested in [24] and further results were presented for this type of preconditioning in [40]. For constant viscosity case, there have been many studies for different choices of the preconditioners for K and S blocks; see [13, 14, 39, 42, 45]. For the discontinuous viscosity case, on the other hand, Olshanskii and Reusken [36, 37] studied the performance of p-Minres with a new PMM preconditioner.

The remainder of the paper is structured as follows. In Section 15.2, we describe p-Uzawa and p-Minres solvers. In Section 15.3, we comparatively study the performance of the AGKS preconditioner against MG used under the above solvers. We highlight important aspects of robust preconditioning and draw some conclusions in Section 15.4.

15.2 Solver Methods

The LBB stability of Stokes discretizations has been extensively studied due to utilization of weak formulations to solve (15.1). We are interested in the LBB stability in the case of high-contrast coefficients. In [36], the LBB stability was proved only for the case $0 < \nu \leq 1$. Later, in [37] this restriction was eliminated and the results were extended to cover general viscosity, thereby, immediately establishing the LBB stability of the discretization under consideration as the following:

$$\sup_{u_h \in V_h} \frac{(\operatorname{div} u_h, p_h)}{\|u_h\|_V} \geq c_{LBB} \, \|p_h\|_Q, \quad p_h \in Q_h, \tag{15.8}$$

The associated spaces and weighted norms are defined as follows:

$$\begin{aligned} V &:= [H_0^1(\Omega)]^d, \\ Q &:= \left\{ p \in L^2(\Omega) : (\nu^{-1} p, 1) = 0 \right\}, \\ \|u\|_V &:= (\nu \nabla u, \nabla v)^{\frac{1}{2}}, \quad u \in V, \\ \|p\|_Q &:= (\nu^{-1} p, p)^{\frac{1}{2}}, \quad p \in Q. \end{aligned}$$

Here $V_h \subset V$ and $Q_h \subset Q$ are finite element spaces that are LBB stable. To be precise, we utilize the $Q2$-$Q1$ (the so-called Taylor-Hood finite element) discretization for numerical experiments in Section 15.3.

There are many solution methods for the indefinite saddle point problem (15.3). We concentrate on two different solver methods: the p-Uzawa and p-Minres. We test the performance the AGKS preconditioner with these solver methods. First, we establish two spectral equivalences: between the velocity stiffness matrix K and the AGKS preconditioner and between the Schur complement matrix S and the scaled PMM. Note that the constant c_{LBB} in (15.8) is directly used for the spectral equivalence of S in the following.

Lemma 15.1. *Let $\hat{K}$ and $\hat{S}$ denote the AGKS preconditioner and the scaled PMM. Then, for sufficiently large m, the following spectral equivalences hold:*

(a)

$$(1 - cm^{-1/2})(\hat{K}u, u) \leq (Ku, u) \leq (1 + cm^{-1/2})(\hat{K}u, u), \tag{15.9}$$

for some constant c independent of m.

(b)

$$c_{LBB}^2 (\hat{S}p, p)_Q \leq (Sp, p) \leq d(\hat{S}p, p)_Q, \tag{15.10}$$

where c_{LBB} is the constant in (15.8) which is independent of m and h.

Proof. One can extract a symmetric positive semidefinite matrix $\mathscr{N}_{HH}^*$ with a rank one kernel from K_{HH}^* in (15.5). $\mathscr{N}_{HH}^*$ is the so-called Neumann matrix and the extraction leads to the following decomposition:

$$K_{HH}^*(m) = m\mathscr{N}_{HH}^* + \Delta.$$

Δ corresponds to the coupling between the DOF in Ω_L and on the boundary of Ω_H. Since $\ker(\mathscr{N}_{HH}^*)$ has rank one, $\mathscr{N}_{HH}^*$ has a simple zero eigenvalue and the below spectral decomposition holds with $\lambda_i > 0, \quad i = 1, \ldots, n_H - 1$ where n_H denotes the order of $\mathscr{N}_{HH}^*$:

$$Z^t \mathscr{N}_{HH}^* Z = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n_H - 1}, 0).$$

Although the eigenvectors in the columns of Z and the eigenvalues λ_i can change according to the underlying discretization, there is always one simple zero eigenvalue and its corresponding constant eigenvector independent of the discretization. This is a direct consequence of the diffusion operator corresponding to a Neumann problem. Therefore, the spectral equivalence established for the $P1$ finite element in [2, Thm. 1] extends to $Q2$ and $Q1$ discretizations, thereby, completing the proof of part (a) for K^*. The spectral equivalence of K easily follows from that of K^* because of the decomposition in (15.4).

The proof of (b) follows from [37, Thm. 6]. □

15.2.1 The Preconditioned Uzawa Solver

The Uzawa algorithm is a classical solution method which involves block factorization with forward and backward substitution. Here, we use the preconditioned inexact Uzawa method described in [12] and [39]. The system (15.3) can be block factorized as follows:

$$\begin{bmatrix} K(m) & 0 \\ B & -I \end{bmatrix} \begin{bmatrix} I & K(m)^{-1}B^t \\ 0 & S(m) \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{15.11}$$

Let (u^k, p^k) be a given approximation of the solution (u, p). Using the block factorization (15.11) combined with a preconditioned Richardson iteration, one obtains:

$$\begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix} = \begin{bmatrix} u^k \\ p^k \end{bmatrix} + \begin{bmatrix} I & -K^{-1}B^tS^{-1} \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} K^{-1} & 0 \\ BK^{-1} & -I \end{bmatrix} \left(\begin{bmatrix} f \\ 0 \end{bmatrix} - \mathscr{A} \begin{bmatrix} u^k \\ p^k \end{bmatrix} \right). \tag{15.12}$$

This leads to the following iterative method:

$$u^{k+1} = u^k + w^k - \hat{K}^{-1}B^t z^k, \tag{15.13a}$$

$$p^{k+1} = p^k + z^k, \tag{15.13b}$$

where $w^k := \hat{K}^{-1} r_1^k$, $r_1^k := f - Ku^k - B^t p^k$, and $z^k := \hat{S}B(w^k + u^k)$. Here, $\hat{K}$ and $\hat{S}$ are the AGKS and PMM preconditioners for K and S, respectively. Computing z^k involves ℓ iterations of pCG. In this computation, since the assembly of S is prohibitively expensive, first we replace it by $\tilde{S}$. Then, we utilize the preconditioner $\hat{K}$ for K and $\hat{S}$ for $\tilde{S}$ where the explicit formula is given by:

$$\tilde{S} := B\hat{K}^{-1}B^t. \tag{15.14}$$

Thus, the total number of applications of $\hat{K}^{-1}$ in (15.13a) and (15.13b) becomes $\ell + 2$. We refer the outer-solve (one Uzawa iteration) as steps (15.13a) and (15.13b) combined. In particular, we call the the computation of z^k as an S-solve; see Table 15.1. The stopping criterion of the S-solve plays an important for the efficiency of the Uzawa method and it is affected by the accuracy of $\hat{K}$; see the analysis in [39, Sec. 4]. When the AGKS preconditioner is used for velocity stiffness matrix, the stopping criterion of the S-solve is determined as follows:

Let r_p^i be the residual of the S-solve at iteration i. Then, we abort the iteration when $\frac{\|r_p^i\|}{\|r_p^0\|} \leq \delta_{tol}$ where

- $\delta_{tol} = 0.5$ or
- maximum iteration reaches 4.

15.2.2 The Preconditioned Minres Solver

The p-Minres is a popular iterative method applied to the system (15.3). Let $v := \begin{bmatrix} u \\ p \end{bmatrix}$. With the given initial guess $v^0 := \begin{bmatrix} u^0 \\ p^0 \end{bmatrix}$ where $p^0 \in e^{\perp_Q}$ and with the corresponding error $r^0 := v - v^0$, the p-Minres solver computes:

$$v^k = \underset{v \in v^0 + \mathscr{K}^k(\mathscr{B}^{-1}\mathscr{A}, \tilde{r}^0)}{\operatorname{argmin}} \|\mathscr{B}^{-1}\left(\begin{bmatrix} f \\ 0 \end{bmatrix} - \mathscr{A}\, v\right)\|.$$

Here, $\tilde{r}^0 = \mathscr{B}^{-1} r^0$ and $\mathscr{K}^k = span\{\tilde{r}^0, \mathscr{B}^{-1}\mathscr{A}\tilde{r}^0, \dots (\mathscr{B}^{-1}\mathscr{A})^k \tilde{r}^0\}$, and the preconditioner has the following block diagonal structure:

$$\mathscr{B} = \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{S} \end{bmatrix}, \tag{15.15}$$

where $\hat{K}$ and $\hat{S}$ are the preconditioners for K and S, respectively. In each step of the p-Minres solver the above preconditioner is applied in the following fashion: for the K-block one application of $\hat{K}$ and for the S-block several applications of pCG to the $\tilde{S}$-system with $\hat{S}$ as the preconditioner. Here, $\tilde{S} = B\hat{K}^{-1}B^t$ stands for the approximation of S. Since S is replaced by $\tilde{S}$, this turns the p-Minres algorithm to an inexact one; see the inexactness discussion in Section 15.3.2. The p-Minres iterations are called outer-solve whereas the pCG solve for the $\tilde{S}$-system is called inner-solve.

The convergence rate of the p-Minres method depends on the condition number of the preconditioned matrix, $\mathscr{B}^{-1}\mathscr{A}$. Combining the spectral equivalences given in (15.9) and (15.10) with the well-known condition number estimate [8], we obtain:

$$\kappa_{\mathscr{B}}(\mathscr{B}^{-1}\mathscr{A}) \leq \frac{\max\{(1 + cm^{-1/2}), d\}}{\min\{(1 - cm^{-1/2}), c_{LBB}^2\}}$$

It immediately follows that the convergence rate of the p-Minres method is independent of m asymptotically.

15.3 Numerical Experiments

The goal of the numerical experiments is to compare the performance of the AGKS and MG preconditioners by using two different solvers: p-Uzawa and p-Minres. We use a four-level hierarchy in which the numbers of DOF, N_1, N_2, N_3, and N_4, are $659, 2467, 9539$, and 37507 from coarsest to finest level. We consider cavity flow with enclosed boundary conditions with right hand side functions $f = 1$ and $g = 0$ on a 2D domain $[-1,1] \times [-1,1]$.

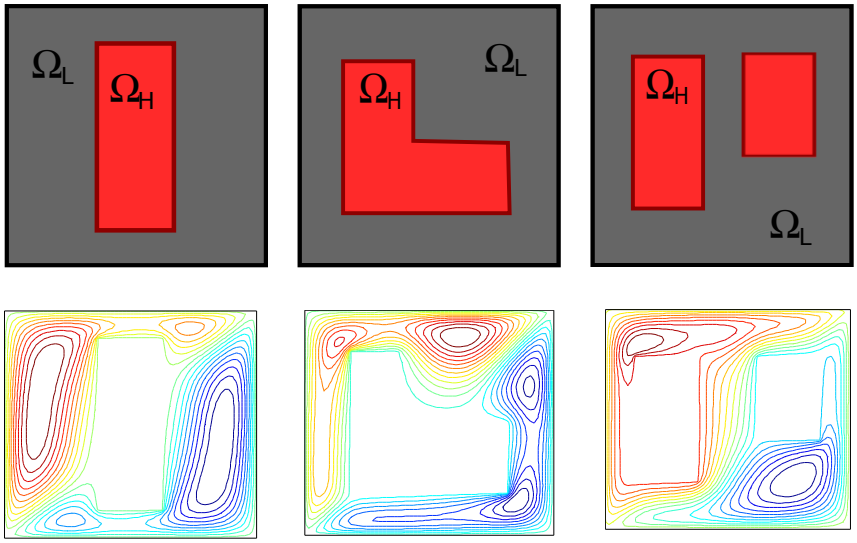

Fig. 15.2: The streamline plot of the high-contrast Stokes equation for three different high-viscosity island configurations; (left) rectangular, (middle) L-shaped, and (right) two disconnected islands.

For discretization, we use the Q_2-Q_1 (the Taylor-Hood) stable finite elements and stabilized Q_1-Q_1 finite elements for the velocity-pressure pair. We consider the case of a single island (viscous inclusion) located at the region $[-1/4,1/4]\times[-3/4,3/4]$. For an extension, we also consider the cases of L-shaped island and two disconnected islands; see Figure 15.2. The observation about these cases are given in Section 15.4. The implementation of discretization is based on ifiss3.1 software provided in [41]; also see [21]. The AGKS preconditioner implementation is based on our implementation in [2, 6, 7]. The implementation of the MG preconditioner is derived from the one in [1]. We employ a V(1,1)-cycle, with point Gauss-Seidel (GS) smoother. A direct solver is used for the coarsest level. For each level of refinement, we present the number of iteration and average reduction factor corresponding to each solve (outer-solve and S-solve; outer-solve and inner-solve for p-Uzawa and p-Minres iterations, respectively). In the tables, N stand for the number of DOF in $\mathscr{A}$ for the outer-solves and the number of DOF in S for the S- and inner-solves. We enforce an iteration bound of 200. If the method seems to converge slightly beyond this bound, we denote it by $*$. A zero initial guess is used. The numerical experiments were performed on a dual core Macbook Pro, running at 2.4 GHz with 4GB RAM.

In analyzing m-robustness, we observe a special feature. The iteration count remains fixed when m becomes larger than a certain threshold value. We define the notion *asymptotic regime* to indicate m values bigger than this threshold. Identifying an asymptotic regime is desirable because it immediately indicates m-robustness.

Table 15.1: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, MG.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**13**,0.546	**15**,0.594	**15**,0.517	**17**,0.517	**19**,0.506	**19**,0.596	**19**,0.589	**19**,0.588	**22**,0.588	**22**,0.589
2467	**13**,0.516	**17**,0.531	**17**,0.552	**18**,0.456	**20**,0.456	**21**,0.455	**21**,0.455	**21**,0.455	**21**,0.456	**21**,0.460
9539	**18**,0.345	**20**,0.460	**20**,0.487	**23**,0.491	**25**,0.492	**26**,0.491	**27**,0.683	**28**,0.677	**31**,0.678	**32**,0.698
37507	**13**,0.371	**23**,0.476	**23**,0.509	**26**,0.508	**27**,0.503	**38**,0.502	**35**,0.500	**40**,0.499	**48**,0.800	**50**,0.825
					S-solve					
81	**2**,0.797	**3**,0.703	**2**,0.715	**2**,0.726	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729	**2**,0.729	**2**,0.729
289	**4**,0.899	**4**,0.903	**4**,0.912	**4**,0.915	**4**,0.915	**4**,0.915	**4**,0.915	**4**,0.915	**4**,0.915	**4**,0.915
1089	**1**,0.997	**2**,0.802	**3**,0.914	**4**,0.919	**4**,0.920	**4**,0.920	**4**,0.920	**3**,0.920	**3**,0.920	**3**,0.920
4225	**1**,0.995	**1**,0.800	**1**,0.913	**1**,0.920	**3**,0.920	**2**,0.921	**4**,0.981	**3**,0.921	**4**,0.941	**3**,0.921

Table 15.2: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, AGKS.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**24**,0.546	**15**,0.394	**14**,0.417	**14**,0.417	**14**,0.406	**14**,0.396	**14**,0.389	**14**,0.388	**14**,0.388	**14**,0.389
2467	**38**,0.316	**21**,0.431	**18**,0.452	**19**,0.456	**18**,0.456	**18**,0.455	**18**,0.455	**18**,0.455	**18**,0.456	**18**,0.460
9539	**47**,0.745	**31**,0.660	**16**,0.487	**16**,0.491	**15**,0.492	**15**,0.491	**15**,0.483	**15**,0.477	**15**,0.478	**15**,0.480
37507	**70**,0.871	**50**,0.476	**17**,0.509	**16**,0.508	**15**,0.503	**15**,0.502	**15**,0.500	**15**,0.499	**15**,0.500	**15**,0.501
					S-solve					
81	**3**,0.420	**2**,0.495	**3**,0.420	**3**,0.427	**3**,0.408	**3**,0.403	**3**,0.401	**3**,0.403	**3**,0.403	**3**,0.403
289	**3**,0.420	**3**,0.495	**3**,0.420	**3**,0.427	**3**,0.408	**3**,0.403	**3**,0.401	**3**,0.403	**3**,0.403	**3**,0.403
1089	**1**,0.620	**1**,0.695	**3**,0.620	**1**,0.627	**1**,0.608	**1**,0.603	**1**,0.601	**1**,0.603	**1**,0.603	**1**,0.603
4225	**1**,0.620	**1**,0.695	**3**,0.620	**1**,0.627	**1**,0.608	**1**,0.603	**1**,0.601	**1**,0.603	**1**,0.603	**1**,0.603

15.3.1 The Preconditioned Uzawa Solver

We use pCG solver with scaled PMM as a preconditioner, 0.5 as tolerance and 4 as maximum number of iterations, for the S-system in each iteration of p-Uzawa. The tolerance for the outer-solve is set to be 5×10^{-6}. We report the performance of the p-Uzawa solver applied to a rectangular and skewed mesh with Q_2-Q_1 discretization. We observe that the p-Uzawa method is m-robust as long as the optimal stopping criterion is used for the S-solve; see Tables 15.1- 15.6. The performances of the AGKS and MG preconditioners are observed as follows. When the MG preconditioner is used, the p-Uzawa solver loses m- and h-robustness and the iteration count increases dramatically when the mesh aspect ratio or the island configuration changes; see Tables 15.1, 15.3, and 15.5. Especially for viscosity values larger than 10^5, we further observe that the iteration number of pCG method for the S-solve, denoted by ℓ, reaches the maximum iteration count 4. Since the MG preconditioner is applied $\ell+2$ times at each iteration of the outer solve, we illustrate how this results in an unreasonable number of applications of the MG preconditioner; see Figure 15.3. For instance in Table 15.1, for the case of $m = 10^8$, we have $\ell = 4$. Therefore, in each outer-iteration, we apply the MG preconditioner $\ell+2 = 6$ times. At level $= 4$, since the total number of MG application is the product of the outer-solve count with $\ell+2$, it becomes $48 \times 6 = 288$. The iteration increases even more rapidly as we refine the mesh. Therefore, the loss of h-robustness sets a major drawback as larger size problems are considered.

Table 15.3: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, skewed mesh, MG.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**16**,0.477	**18**,0.488	**70**,0.817	$*$,0.977	$*$,0.996	$*$,0.996	$*$,0.989	$*$,0.988	$*$,0.988	$*$,0.989
2467	**18**,0.616	**21**,0.631	**24**,0.652	**34**,0.706	**80**,0.856	$*$,0.955	$*$,0.955	$*$,0.965	$*$,0.956	$*$,0.990
9539	**19**,0.515	**23**,0.570	**32**,0.687	$*$,0.991	$*$,0.952	$*$,0.991	$*$,0.983	$*$,0.687	$*$,0.978	$*$,0.998
37507	**17**,0.471	**27**,0.576	**27**,0.569	**72**,0.808	**97**,0.883	$*$,0.962	$*$,0.990	$*$,0.999	$*$,0.980	$*$,0.985
					S-solve					
81	**2**,0.797	**3**,0.703	**2**,0.715	**2**,0.726	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729
289	**4**,0.899	**4**,0.903	**4**,0.912	**4**,0.915	**4**,0.915	**3**,0.915	**4**,0.915	**4**,0.915	**4**,0.915	**4**,0.915
1089	**1**,0.997	**2**,0.802	**3**,0.914	**4**,0.919	**4**,0.920	**3**,0.920	**4**,0.920	**3**,0.920	**3**,0.920	**3**,0.920
4225	**2**,0.995	**1**,0.800	**1**,0.913	**2**,0.920	**3**,0.920	**3**,0.921	**4**,0.981	**3**,0.921	**3**,0.941	**3**,0.921

Table 15.4: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, skewed mesh, AGKS.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**33**,0.695	**23**,0.580	**21**,0.556	**21**,0.553	**22**,0.564	**22**,0.564	**22**,0.564	**21**,0.545	**19**,0.529	**25**,0.603
2467	**57**,0.805	**29**,0.649	**23**,0.592	**27**,0.632	**29**,0.653	**33**,0.684	**35**,0.700	**24**,0.639	**32**,0.720	**33**,0.693
9539	**72**,0.875	**39**,0.733	**31**,0.666	**38**,0.728	**32**,0.682	**40**,0.737	**33**,0.691	**31**,0.667	**31**,0.701	**31**,0.708
37507	**91**,0.921	**59**,0.811	**31**,0.679	**36**,0.708	**28**,0.633	**32**,0.762	**30**,0.700	**29**,0.669	**31**,0.698	**31**,0.701
					S-solve					
81	**2**,0.750	**2**,0.722	**2**,0.769	**2**,0.806	**2**,0.915	**2**,0.918	**2**,0.918	**2**,0.922	**5**,0.897	**5**,0.916
289	**3**,0.814	**3**,0.791	**2**,0.800	**3**,0.727	**3**,0.708	**3**,0.703	**4**,0.711	**3**,0.723	**2**,0.703	**3**,0.703
1089	**2**,0.700	**3**,0.695	**3**,0.720	**2**,0.727	**2**,0.708	**2**,0.688	**2**,0.701	**2**,0.713	**2**,0.693	**2**,0.703
4225	**3**,0.720	**2**,0.695	**3**,0.720	**2**,0.697	**2**,0.688	**2**,0.603	**2**,0.701	**2**,0.703	**2**,0.723	**2**,0.713

On the other hand, the AGKS preconditioner maintains m- and h-robustness simultaneously. Asymptotically, only one iteration of pCG is sufficient to obtain an accurate S-solve for a rectangular mesh; see Table 15.2. When we do the above calculation, we find that for a rectangular mesh, the total number of AGKS applications is $15 \times (1+2) = 45$. Since this application count remains fixed as the mesh is refined, we infer the h-robustness of the AGKS preconditioner; see Figure 15.3. When the mesh aspect ratio or the island configuration changes, number of pCG iterations required to have an accurate S-solve becomes 2 or 3. However, this is reasonable since the outer-solve maintains h- an m- robustness; see Tables 15.4, and 15.6. Hence, the AGKS preconditioner will acceleratedly outperform the MG preconditioner as more mesh refinements are introduced regardless of the island or mesh configuration.

15.3.2 The Preconditioned Minres Solver

We notice that the p-Minres has not been the solver of choice for high-contrast problems due to its unfavorable performance with PMM for the S-system; see [36]. We have taken a novel approach for the S system. First, we replace S by $\tilde{S} = B\hat{K}^{-1}B^t$ where $\hat{K}^{-1}$ step is one application of the AGKS preconditioner. This makes the solver method *inexact*. Then, we solve $\tilde{S}$ system by using a pCG solver with scaled PMM preconditioner with tolerance 0.05 with a maximum of 20 iterations. The pCG and p-Minres solution steps are called the inner- and outer-solve, respectively.

Table 15.5: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, L-shaped island, MG.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**12**,0.377	**16**,0.478	**27**,0.617	**67**,0.717	**97**,0.817	∗,0.976	∗,0.969	∗,0.980	∗,0.988	∗,0.989
2467	**13**,0.316	**17**,0.431	**17**,0.452	**20**,0.546	**80**,0.836	∗,0.951	∗,0.965	∗,0.962	∗,0.976	∗,0.992
9539	**18**,0.525	**24**,0.575	**25**,0.607	**29**,0.657	**87**,0.887	∗,0.981	∗,0.978	∗,0.987	∗,0.988	∗,0.998
37507	**18**,0.491	**27**,0.667	**27**,0.649	**49**,0.778	**50**,0.793	∗,0.962	∗,0.995	∗,0.993	∗,0.998	∗,0.985
					S-solve					
81	**2**,0.797	**3**,0.703	**2**,0.715	**2**,0.726	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729	**3**,0.729
289	**4**,0.899	**4**,0.903	**4**,0.912	**4**,0.915	**4**,0.915	**3**,0.915	**4**,0.915	**4**,0.915	**4**,0.915	**4**,0.915
1089	**1**,0.997	**2**,0.802	**3**,0.914	**4**,0.919	**4**,0.920	**3**,0.920	**4**,0.920	**3**,0.920	**3**,0.920	**3**,0.920
4225	**2**,0.995	**1**,0.800	**1**,0.913	**2**,0.920	**3**,0.920	**3**,0.921	**4**,0.981	**3**,0.921	**3**,0.941	**3**,0.921

Our approach for the S-system is similar to the one we take in the p-Uzawa solver. But, notice that now the inner solver requires more accuracy in order to guarantee a convergent p-Minres solver.

Table 15.6: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, L-shaped island, AGKS.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**30**,0.660	**23**,0.583	**22**,0.568	**23**,0.580	**24**,0.596	**26**,0.616	**25**,0.611	**27**,0.628	**25**,0.613	**25**,0.609
2467	**52**,0.791	**27**,0.631	**18**,0.503	**16**,0.454	**19**,0.525	**18**,0.509	**19**,0.527	**21**,0.581	**21**,0.556	**21**,0.560
9539	**68**,0.875	**50**,0.780	**24**,0.591	**22**,0.591	**23**,0.582	**28**,0.611	**28**,0.653	**29**,0.687	**31**,0.687	**32**,0.718
37507	**73**,0.871	**53**,0.796	**25**,0.599	**24**,0.510	**26**,0.524	**34**,0.512	**35**,0.502	**33**,0.490	**42**,0.804	**42**,0.815
					S-solve					
81	**2**,0.787	**3**,0.703	**2**,0.717	**2**,0.726	**3**,0.729	**3**,0.725	**3**,0.729	**3**,0.729	**2**,0.739	**2**,0.729
289	**4**,0.895	**4**,0.904	**4**,0.922	**4**,0.911	**4**,0.935	**4**,0.921	**4**,0.915	**4**,0.915	**4**,0.912	**4**,0.915
1089	**2**,0.997	**3**,0.802	**3**,0.914	**3**,0.921	**4**,0.920	**4**,0.920	**4**,0.919	**3**,0.920	**3**,0.920	**3**,0.920
4225	**2**,0.995	**2**,0.810	**2**,0.913	**2**,0.920	**3**,0.919	**2**,0.921	**3**,0.981	**3**,0.921	**4**,0.941	**3**,0.921

Table 15.7: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, MG.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**15**,0.546	**15**,0.594	**18**,0.517	**19**,0.517	**19**,0.506	**19**,0.596	**20**,0.589	**20**,0.588	**23**,0.588	**24**,0.589
2467	**20**,0.516	**19**,0.531	**21**,0.552	**24**,0.456	**23**,0.456	**24**,0.455	**25**,0.455	**28**,0.455	**29**,0.456	**30**,0.460
9539	**21**,0.345	**19**,0.460	**24**,0.487	**24**,0.491	**24**,0.492	**24**,0.491	**25**,0.683	**26**,0.677	**28**,0.678	**32**,0.698
37507	**21**,0.371	**21**,0.476	**23**,0.509	**26**,0.508	**30**,0.503	**26**,0.502	**29**,0.500	**31**,0.499	**34**,0.800	**36**,0.825
					inner-solve					
81	**6**,0.497	**7**,0.623	**7**,0.655	**7**,0.666	**7**,0.649	**7**,0.659	**7**,0.659	**7**,0.659	**7**,0.660	**7**,0.661
289	**8**,0.699	**9**,0.703	**9**,0.713	**9**,0.720	**9**,0.715	**9**,0.717	**9**,0.713	**9**,0.721	**9**,0.735	**9**,0.735
1089	**9**,0.747	**11**,0.752	**11**,0.744	**11**,0.749	**11**,0.750	**11**,0.760	**11**,0.759	**11**,0.761	**11**,0.762	**11**,0.760
4225	**12**,0.795	**13**,0.801	**13**,0.813	**13**,0.810	**13**,0.811	**13**,0.801	**13**,0.803	**13**,0.805	**13**,0.808	**13**,0.811

Table 15.8: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, AGKS.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**29**,0.546	**23**,0.394	**18**,0.417	**16**,0.417	**18**,0.406	**16**,0.396	**16**,0.389	**18**,0.388	**20**,0.388	**20**,0.389
2467	**40**,0.316	**30**,0.431	**17**,0.452	**17**,0.456	**16**,0.456	**16**,0.455	**16**,0.455	**19**,0.455	**19**,0.456	**19**,0.460
9539	**50**,0.745	**45**,0.660	**20**,0.487	**20**,0.491	**19**,0.492	**16**,0.491	**16**,0.483	**20**,0.477	**20**,0.478	**20**,0.608
37507	**70**,0.871	**52**,0.476	**22**,0.509	**20**,0.508	**19**,0.503	**16**,0.502	**16**,0.500	**20**,0.499	**20**,0.500	**20**,0.525
					inner-solve					
81	**20**,0.797	**20**,0.703	**5**,0.585	**5**,0.576	**5**,0.579	**5**,0.529	**5**,0.569	**5**,0.548	**5**,0.567	**5**,0.554
289	**20**,0.763	**20**,0.694	**5**,0.580	**5**,0.572	**5**,0.582	**5**,0.532	**5**,0.556	**5**,0.552	**5**,0.571	**5**,0.548
1089	**20**,0.773	**20**,0.714	**5**,0.580	**5**,0.575	**5**,0.578	**5**,0.532	**5**,0.561	**5**,0.555	**5**,0.562	**5**,0.556
4225	**20**,0.768	**20**,0.701	**5**,0.583	**5**,0.573	**5**,0.576	**5**,0.530	**5**,0.561	**5**,0.550	**5**,0.548	**5**,0.552

As in the p-Uzawa case, the effectiveness of the AGKS preconditioner has been confirmed as it maintains both the m- and h-robustness whereas MG suffers from the loss of both; see Tables 15.7 - 15.12. Furthermore, we observe that the choice of $\hat{K}^{-1}$–an application of either MG or AGKS–in the inner-solve dramatically affects the performance inner-solve. Specifically, the scaled PMM preconditioner is m-robust, but not h-robust for the inner-solve with MG, whereas it is both m- and h-robust for inner-solve with AGKS regardless of the mesh aspect ratio or island configuration.

15.4 Conclusion

We provide several concluding remarks on the performance of the AGKS preconditioner under two different solvers. For p-Uzawa and p-Minres solvers, we report numerical results for only $Q2$-$Q1$ discretization on a rectangular or skewed mesh with a single square shaped or L-shaped island.

The p-Uzawa solver turns out to be the best choice since AGKS preserves both m- and h-robustness regardless of the discretization type or deterioration in the aspect ratio of the mesh. The change in one of the above only causes increase in the number of iterations, but qualitatively m- and h-robustness are maintained. Moreover, we observe that the asymptotic regime of the p-Uzawa solver starts with the m value 10^3; see left-bottom in Figure 15.3. As island configuration changes, the number of iterations of both K- and S-solve slightly increases. In addition to that, as the discretization changes, the m-robustness of PMM for S-solve is lost. The asymptotic regime of the p-Uzawa solver becomes $m \geq 10^7$; see Tables 15.4 and 15.6.

The AGKS preconditioner under the p-Minres solver also maintains both m- and h-robustness as the discretization, the aspect ratio of the mesh, or the island configuration change. However, the number of iterations in the p-Minres solver increases dramatically when the mesh is skewed. Compared to p-Uzawa, one needs a more accurate inner-solve for a convergent p-Minres. In addition, the asymptotic regime of p-Minres solver is $m \geq 10^7$. Combining these three features, p-Minres becomes less desirable compared to p-Uzawa; see bottom-right and top-right in Figure 15.3. However, this solver is potentially useful for large size problems as the AGKS preconditioner maintains h-robustness.

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Table 15.9: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, skewed mesh, MG.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**18**,0.547	**19**,0.578	**19**,0.582	**28**,0.657	**43**,0.767	**88**,0.897	$*$,0.999	$*$,0.999	$*$,0.998	$*$,0.999
2467	**20**,0.556	**21**,0.581	**26**,0.622	**32**,0.706	**46**,0.786	**92**,0.905	$*$,0.942	$*$,0.971	$*$,0.989	$*$,0.995
9539	**20**,0.545	**25**,0.578	**29**,0.677	**43**,0.757	**57**,0.817	**97**,0.917	$*$,0.978	$*$,0.987	$*$,0.988	$*$,0.998
37507	**20**,0.561	**27**,0.657	**31**,0.679	**49**,0.783	**70**,0.813	**119**,0.883	$*$,0.991	$*$,0.998	$*$,0.998	$*$,0.999
					inner-solve					
81	**6**,0.499	**9**,0.633	**9**,0.675	**9**,0.686	**9**,0.679	**9**,0.679	**9**,0.679	**9**,0.679	**9**,0.680	**9**,0.681
289	**8**,0.729	**10**,0.753	**10**,0.753	**10**,0.760	**10**,0.755	**10**,0.757	**10**,0.753	**10**,0.761	**10**,0.775	**10**,0.775
1089	**9**,0.787	**13**,0.792	**13**,0.784	**13**,0.789	**13**,0.790	**13**,0.810	**13**,0.819	**13**,0.811	**13**,0.822	**13**,0.820
4225	**12**,0.825	**15**,0.851	**15**,0.865	**15**,0.860	**15**,0.861	**15**,0.851	**15**,0.853	**15**,0.855	**15**,0.858	**15**,0.861

Table 15.10: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, skewed mesh, AGKS.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**33**,0.694	**26**,0.647	**26**,0.644	**27**,0.649	**27**,0.654	**29**,0.663	**29**,0.665	**31**,0.679	**39**,0.753	**38**,0.785
2467	**59**,0.816	**33**,0.713	**29**,0.674	**29**,0.690	**37**,0.733	**36**,0.740	**36**,0.761	**46**,0.773	**49**,0.792	**48**,0.783
9539	**75**,0.875	**44**,0.772	**34**,0.702	**34**,0.711	**49**,0.782	**58**,0.814	**60**,0.820	**60**,0.840	**70**,0.893	**72**,0.902
37507	**90**,0.851	**55**,0.807	**38**,0.729	**39**,0.733	**50**,0.783	**61**,0.823	**68**,0.840	**68**,0.858	**75**,0.903	**72**,0.902
					inner-solve					
81	**25**,0.817	**23**,0.813	**15**,0.787	**15**,0.776	**15**,0.779	**15**,0.729	**15**,0.769	**15**,0.748	**15**,0.767	**15**,0.754
289	**20**,0.763	**20**,0.694	**15**,0.780	**15**,0.772	**15**,0.782	**15**,0.732	**15**,0.776	**15**,0.772	**15**,0.771	**15**,0.748
1089	**20**,0.773	**20**,0.714	**15**,0.780	**15**,0.777	**15**,0.778	**15**,0.732	**15**,0.761	**15**,0.755	**15**,0.762	**15**,0.756
4225	**20**,0.768	**20**,0.701	**15**,0.783	**15**,0.773	**15**,0.776	**15**,0.730	**15**,0.761	**16**,0.770	**15**,0.748	**15**,0.772

Table 15.11: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, L-shaped island, MG.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**12**,0.377	**16**,0.478	**27**,0.617	**67**,0.717	**97**,0.817	∗,0.976	∗,0.969	∗,0.980	∗,0.988	∗,0.989
2467	**13**,0.316	**17**,0.431	**17**,0.452	**20**,0.546	**80**,0.836	∗,0.951	∗,0.965	∗,0.962	∗,0.976	∗,0.992
9539	**18**,0.525	**24**,0.575	**25**,0.607	**29**,0.657	**87**,0.887	∗,0.981	∗,0.978	∗,0.987	∗,0.988	∗,0.998
37507	**18**,0.491	**27**,0.667	**27**,0.649	**49**,0.778	**50**,0.793	∗,0.962	∗,0.995	∗,0.993	∗,0.998	∗,0.985
					inner-solve					
81	**6**,0.497	**7**,0.625	**7**,0.655	**7**,0.666	**7**,0.649	**7**,0.659	**7**,0.659	**8**,0.669	**8**,0.670	**8**,0.671
289	**8**,0.699	**9**,0.703	**9**,0.713	**9**,0.720	**9**,0.715	**9**,0.717	**9**,0.713	**10**,0.741	**10**,0.745	**10**,0.745
1089	**9**,0.747	**11**,0.752	**11**,0.744	**11**,0.749	**11**,0.750	**11**,0.760	**11**,0.759	**12**,0.765	**12**,0.769	**12**,0.770
4225	**12**,0.795	**13**,0.801	**13**,0.813	**13**,0.810	**13**,0.811	**13**,0.801	**13**,0.803	**14**,0.807	**14**,0.818	**14**,0.819

Table 15.12: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, L-shaped island AGKS.

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					outer-solve					
659	**39**,0.746	**33**,0.694	**32**,0.697	**32**,0.677	**32**,0.706	**32**,0.696	**32**,0.689	**34**,0.718	**34**,0.748	**32**,0.819
2467	**54**,0.816	**41**,0.731	**24**,0.752	**20**,0.656	**18**,0.556	**18**,0.555	**20**,0.585	**21**,0.595	**23**,0.606	**22**,0.760
9539	**60**,0.845	**51**,0.760	**31**,0.687	**23**,0.591	**21**,0.492	**20**,0.591	**22**,0.583	**23**,0.577	**24**,0.578	**23**,0.608
37507	**82**,0.871	**64**,0.476	**35**,0.509	**27**,0.508	**23**,0.503	**21**,0.502	**23**,0.500	**22**,0.499	**23**,0.500	**21**,0.525
					inner-solve					
81	**20**,0.797	**20**,0.703	**6**,0.588	**5**,0.577	**6**,0.581	**6**,0.532	**6**,0.575	**6**,0.553	**6**,0.569	**6**,0.565
289	**20**,0.763	**20**,0.694	**6**,0.582	**6**,0.576	**6**,0.584	**6**,0.536	**6**,0.569	**6**,0.565	**6**,0.574	**6**,0.549
1089	**20**,0.773	**20**,0.714	**6**,0.583	**6**,0.577	**6**,0.579	**6**,0.535	**6**,0.563	**6**,0.667	**6**,0.565	**6**,0.568
4225	**20**,0.768	**20**,0.701	**6**,0.585	**6**,0.576	**6**,0.578	**6**,0.532	**6**,0.560	**6**,0.561	**6**,0.554	**6**,0.561

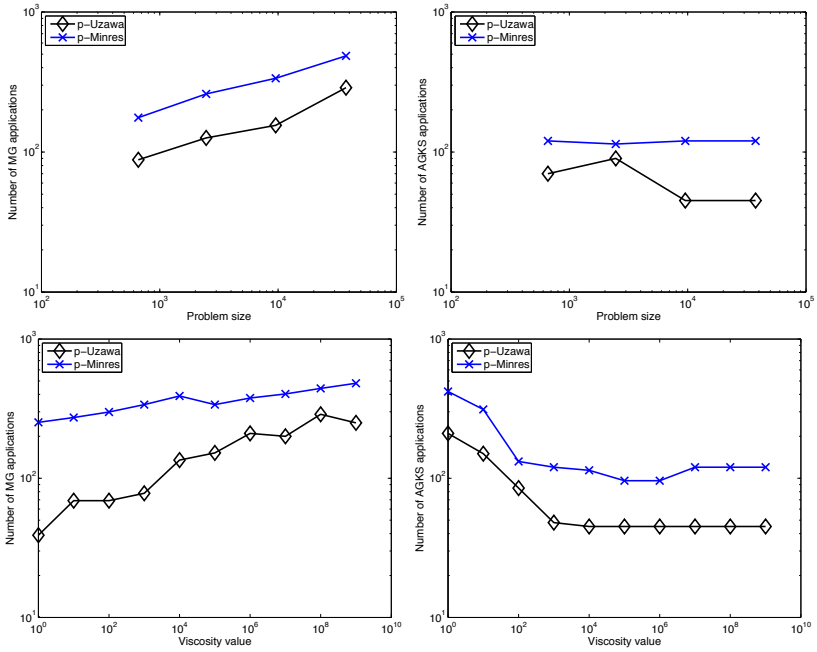

Fig. 15.3: The plot of the number of (top-left) MG applications versus problem size for fixed viscosity value $m = 10^8$, (bottom-left) MG applications versus viscosity value for fixed level $= 4$ (top-right) AGKS applications vs problem size for fixed viscosity value $m = 10^8$, (bottom-right) AGKS applications versus viscosity value for fixed level $= 4$.

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