Comparison of Nonlocal Operators Utilizing Perturbation Analysis

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Abstract. We present a comparative study of integral operators used in nonlocal problems. The size of nonlocality is determined by the parameter δ. The authors recently discovered a way to incorporate local boundary conditions into nonlocal problems. We construct two nonlocal operators which satisfy local homogeneous Neumann boundary conditions. We compare the bulk and boundary behaviors of these two to the operator that enforces nonlocal boundary conditions. We construct approximations to each operator using perturbation expansions in the form of Taylor polynomials by consistently keeping the size of expansion neighborhood equal to δ. In the bulk, we show that one of these two operators exhibits similar behavior with the operator that enforces nonlocal boundary conditions.

1 Introduction

The integral operators under consideration are used, for instance, in peridynamics (PD) [11] and nonlocal diffusion [5,8]. PD is a nonlocal extension of continuum mechanics developed by Silling [11]. PD is based on nonlocal interactions. As a result, nonlocal boundary conditions (BC) are used. The authors recently discovered a way to incorporate local BC into nonlocal theories [1,2,7], in particular into PD.

We present a comparative study of operators used in nonlocal problems. We consider problems in 1D and choose the domain $\Omega = (-1, 1)$. We define the governing operator related to PD by

$$Au(x) := cu(x) - \int_{\Omega} C(x-y)u(y)dy, \quad x \in \Omega, \tag{1}$$

where $C, u \in L^2(\Omega)$ and $c = \int_{\Omega} C(y)dy$. The kernel function $C(x)$ is assumed to be even. An important first choice of $C(x)$ is the *canonical* kernel function

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$\chi_\delta(x)$ whose only role is the representation of the nonlocal neighborhood, called the *horizon*, by a characteristic function. Namely,

$$\chi_\delta(x) := \begin{cases} 1, |x| < \delta \\ 0, \text{otherwise.} \end{cases}$$

The size of nonlocality is determined by δ and we assume $\delta < 1$.

In $\mathbb{R}^d, d \geq 1$, we discovered that the PD governing operator (1) is a bounded function of the classical (Laplace) operator [7]. We generalized this theoretical result to bounded domains [1,2]. The main idea of the generalization is as follows. Building on the theoretical result, we generalized the standard integral based convolution in (1) to an abstract convolution operator which is defined by a Hilbert (complete and orthonormal) basis. This basis is induced by the classical operator with prescribed local BC on bounded domains. The nonlocal operator becomes a function of the classical operator. By prescribing BC to the classical operator, we construct a gateway to incorporate local BC into nonlocal theories.

Through the use of local BC, we plan to solve important elasticity applications which require local BC such as contact, shear, and traction. In addition, we anticipate to eliminate the surface effects which are seen in PD due to employing nonlocal BC. Incorporation of local BC leads to a modification of the original PD governing operator in (1).

The operators $\mathcal{M}$ and $\mathcal{N}$ defined below employ the even part of u. For notational convenience, we denote the orthogonal projections that give the even and odd parts, respectively, of a function by $P_e, P_o : \ L^2(\Omega) \ \rightarrow \ L^2(\Omega)$, whose definitions are

$$P_e u(x) := \frac{u(x) + u(-x)}{2}, \quad P_o u(x) := \frac{u(x) - u(-x)}{2}. \tag{2}$$

In this paper, we present a comparative study of the following three operators. For $x \in \Omega$,

$$\mathcal{L}u(x) := cu(x) - \int_\Omega \chi_\delta(x-y)u(y)dy, \tag{3}$$

$$\mathcal{M}u(x) := cu(x) - \int_\Omega \hat{\chi}_\delta(x-y)P_e u(y)dy, \tag{4}$$

$$\mathcal{N}u(x) := cu(x) - \int_\Omega \hat{\chi}_\delta(|x-y|-1)P_e u(y)dy. \tag{5}$$

Here, we define the extended domain $\hat{\Omega} := (-2, 2)$ and denote the periodic extension of $\chi_\delta(x)|_{x\in\Omega}$ to $\hat{\Omega}$ by $\hat{\chi}_\delta(x)$; see Figure 1. We construct approximations $\widetilde{\mathcal{L}}, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ to each governing operator $\mathcal{L}, \mathcal{M}, \mathcal{N}$ using perturbation expansions. Similar expansions were used by the first author [3,4] and in higher order gradient applications [6,9,10].

2 Operator Definitions

For $x \in (-2,2)$, kernel functions in (3), (4), and (5) are defined as

$$\chi_\delta(x) := \begin{cases} 1, \, x \in (-\delta,\delta) \\ 0, \, \text{otherwise.} \end{cases}$$
$$\hat{\chi}_\delta(x) := \begin{cases} 1, \, x \in (-2,-2+\delta) \cup (-\delta,\delta) \cup (2-\delta,2) \\ 0, \, \text{otherwise.} \end{cases}$$
$$\hat{\chi}_\delta(1-|x|) := \begin{cases} 1, \, x \in (-1-\delta,-1+\delta) \cup (1-\delta,1+\delta) \\ 0, \, \text{otherwise.} \end{cases}$$

The corresponding convolution kernels are

$$\chi_\delta(y-x) := \begin{cases} 1, \, y \in (x-\delta,x+\delta) \\ 0, \, \text{otherwise.} \end{cases}$$
$$\hat{\chi}_\delta(y-x) := \begin{cases} 1, \, y \in (x-2,x-2+\delta) \cup (x-\delta,x+\delta) \cup (x+2-\delta,x+2) \\ 0, \, \text{otherwise.} \end{cases}$$
$$\hat{\chi}_\delta(1-|y-x|) := \begin{cases} 1, \, y \in (x-1-\delta,x-1+\delta) \cup (x+1-\delta,x+1+\delta) \\ 0, \, \text{otherwise.} \end{cases}$$

With a slight abuse of notation, for functions $\hat{\chi}_\delta(\cdot) : \hat{\Omega} \to \mathbb{R}$ and its bivariate version $\hat{\chi}_\delta(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}$, we use the same notation; $\hat{\chi}_\delta(x-y) = \hat{\chi}_\delta(x,y)$. In Figure 1, we depict the support of $\hat{\chi}_\delta(x,y)$.

For integration, we need to consider the following y-ranges

$$\begin{aligned} \Omega_{\mathcal{L}} &:= (-1,1) \cap \{(x-\delta,x+\delta)\}, \\ \Omega_{\mathcal{M}} &:= (-1,1) \cap \{(x-2,x-2+\delta) \cup (x-\delta,x+\delta) \cup (x+2-\delta,x+2)\}, \\ \Omega_{\mathcal{N}} &:= (-1,1) \cap \{(x-1-\delta,x-1+\delta) \cup (x+1-\delta,x+1+\delta)\}. \end{aligned}$$

2.1 Boundary Conditions

The classical operator satisfying homogeneous Neumann BC is given by

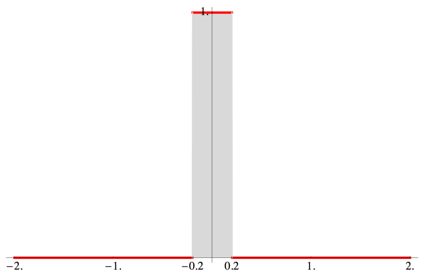
$$A_{\mathtt{N}} u = -\frac{4}{\pi^2} u'',$$

where $'$ denotes the weak derivative and $u \in H_0^2(\Omega)$. $A_{\mathtt{N}}$ has a purely discrete spectrum $\sigma(A_{\mathtt{N}})$ consisting of simple eigenvalues,

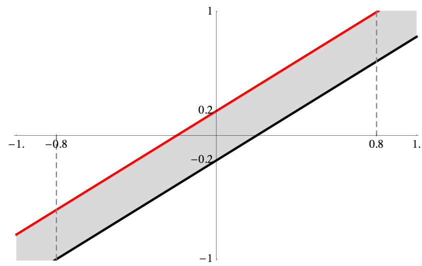
$$\sigma(A_{\mathtt{N}}) = \left\{ k^2 : k \in \mathbb{N} \right\}.$$

A normalized eigenvector corresponding to the eigenvalue k^2 is given by

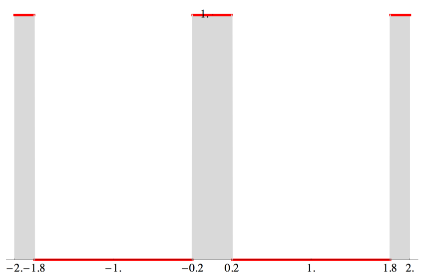
$$e_k^{\mathtt{N}}(x) := \begin{cases} \frac{1}{\sqrt{2}}, & k = 0 \\ \cos\left(\frac{k\pi}{2}(x+1)\right), & k \neq 0, k \in \mathbb{N} \end{cases}.$$



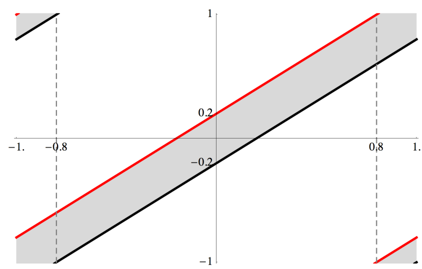
(a) Kernel function $\chi_\delta(x)$.



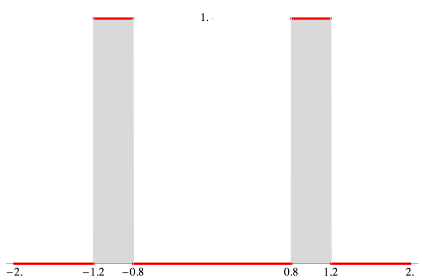
(b) Support of $\chi_\delta(x-y)$.



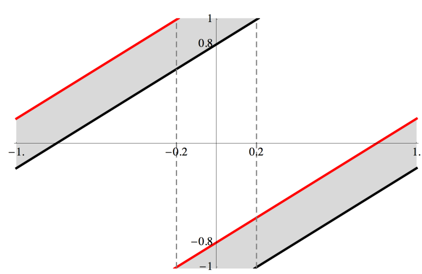
(c) Kernel function $\hat{\chi}_\delta(x)$, periodic extension of $\chi_\delta(x)|_{x\in\Omega}$ to $\hat{\Omega}$.



(d) Support of $\hat{\chi}_\delta(x-y)$.



(e) Kernel function $\hat{\chi}_\delta(1-|x|)$.



(f) Support of $\hat{\chi}_\delta(1-|x-y|)$.

Fig. 1. Kernel functions are obtained by extensions from $\Omega = (-1, 1)$ to $\hat{\Omega} = (-2, 2)$ with $\delta = 0.2$ and their corresponding supports.

The sequence $(e_k^{\mathtt{N}})_{k\in\mathbb{N}}$ is a Hilbert basis of $L^2(\Omega)$. Using this basis, we define the generalized convolution operator on Ω for C, $u \in L^2(\Omega)$ [1,2] as follows

$$\mathcal{C} *_{\mathtt{N}} u(x) := \sum_{k\in\mathbb{N}} \langle e_k^{\mathtt{N}}|C\rangle \, \langle e_k^{\mathtt{N}}|u\rangle \, e_k^{\mathtt{N}}(x), \tag{6}$$

where $\langle\cdot|\cdot\rangle$ denotes the inner product in $L^2(\Omega)$.

We want to obtain an integral representation for (6). For this, we need several ingredients. Let $\widehat{C}(x)$, $x \in (-2,2)$ denote periodic extension of the kernel function $C(x)$, $x \in (-1,1)$. Since $C(x)$ is even, so is $\widehat{C}(x)$. Then, $\widehat{C}(x) = \widehat{C}(|x|)$. The integral representation of $\mathcal{C}*_{\mathtt{N}}$ is based on the following decomposition of $\widehat{C}(|x|)$ based on the "half-wave symmetry."

$$\begin{aligned}\widehat{C}(|x|) \;&=\; \frac{\widehat{C}(|x|)+\widehat{C}(1-|x|)}{2} + \frac{\widehat{C}(|x|)-\widehat{C}(1-|x|)}{2},\\ &=: \widehat{C}_1(x) + \widehat{C}_2(x).\end{aligned}$$

Then, the integral representation of $\mathcal{C}*_{\mathtt{N}}$ in (6) takes the following form [1,2]

$$\mathcal{C} *_{\mathtt{N}} u(x) = \int_\Omega \widehat{C}(|x-y|-1) P_e u(y)dy + \gamma_{N,C} \int_\Omega u(y)dy, \tag{7}$$

where $\gamma_{N,C} := -\frac{\sqrt{2}-1}{2\sqrt{2}} \int_\Omega C_1(y)dy + \frac{\sqrt{2}+1}{2\sqrt{2}} \int_\Omega C_2(y)dy$. Hence,

$$\frac{d}{dx}\mathcal{C} *_{\mathtt{N}} u(x) = \frac{d}{dx}\int_\Omega \widehat{C}(|x-y|-1) P_e u(y)dy.$$

Observe that only the convolution part survives after differentiation. This allows us to induce several governing integral operators that satisfy homogeneous Neumann BC. As a result, we can obtain the operator $\mathcal{N}u$ in (5) with general kernel function $\widehat{C}(|x-y|-1)$

$$\mathcal{N}u(x) := cu(x) - \int_\Omega \widehat{C}(|x-y|-1) P_e u(y)dy. \tag{8}$$

Note that

$$\widehat{C}(|x|-1) = \widehat{C}_1(x) - \widehat{C}_2(x). \tag{9}$$

Using the fact that $\widehat{C}(x)$ is 2-periodic and after some algebraic manipulation, we conclude that both $\widehat{C}_1(x)$ and $\widehat{C}_2(x)$ are 2-periodic. Due to (9), $\widehat{C}(|x|-1)$ is also 2-periodic. Consequently, $\widehat{C}(|x|-1)$ is an even and 2-periodic function, a crucial property that we will also use for constructing the other governing operator; see (13).

Remark 1. $\widehat{C}_1(x)$ and $\widehat{C}_2(x)$ have an additional property of half-wave symmetry. Namely, for every $x \in [0, 1/2]$,

$$C_1(x) = \frac{1}{2}\left[C(x) + C(1-x)\right] = \frac{1}{2}\left[C(|1-x|) + C(1-|1-x|)\right] = C_1(1-x),$$
$$C_2(x) = \frac{1}{2}\left[C(x) - C(1-x)\right] = \frac{1}{2}\left[C(1-|1-x|) - C(|1-x|)\right] = -C_2(1-x).$$

These identities have been used in obtaining the integral representation in (7).

Next, we want to show that the governing operator in (8) satisfies the homogeneous Neumann BC. We begin with rewriting $\mathcal{N}u(x)$ as follows

$$\mathcal{N}u(x) = cu(x) - \int_\Omega (1/2)\left(\widehat{C}(|x-y|-1) + \widehat{C}(|x+y|-1)\right) u(y)]dy.$$

For simplicity, assuming that C is sufficiently smooth and differentiating both sides, we obtain

$$\frac{d}{dx}\mathcal{N}u(x) = cu'(x) - \\ \int_\Omega (1/2)\left(\widehat{C}'(|x-y|-1)\frac{|x-y|}{x-y} + \widehat{C}'(|x+y|-1)\frac{|x+y|}{x+y}\right) u(y)]dy. \quad (10)$$

The case of non-smooth C can be handled by splitting the integral into parts where C is piecewise smooth. Here, $u'(x)$ denotes the initial velocity, and hence, we always assume that it satisfies homogeneous Neumann BC, i.e., $u'(-1) = u'(1) = 0$ because initial values automatically satisfy the given BC. Furthermore, since $\widehat{C}(y)$ is even, $\widehat{C}'(y)$ is odd. Evaluating (10) at $x = -1$ gives

$$\begin{aligned}\frac{d}{dx}\mathcal{N}u(-1) &= cu'(-1) - \int_\Omega (1/2)\left(\widehat{C}'(y)(-1) + \widehat{C}'(-y)(-1)\right) u(y)dy \\ &= cu'(-1) - \int_\Omega (1/2)\left(\widehat{C}'(y)(-1) - \widehat{C}'(y)(-1)\right) u(y)dy \\ &= 0. \end{aligned} \quad (11)$$

Similarly, at $x = 1$, we have

$$\begin{aligned}\frac{d}{dx}\mathcal{N}u(1) &= cu'(1) - \int_\Omega (1/2)\left(\widehat{C}'(-y)(+1) + \widehat{C}'(y)(+1)\right) u(y)dy \\ &= cu'(1) - \int_\Omega (1/2)\left(-\widehat{C}'(y)(+1) + \widehat{C}'(y)(+1)\right) u(y)dy \\ &= 0. \end{aligned} \quad (12)$$

2.2 An Alternative Governing Operator

The main property we exploit in satisfying the BC is the evenness of the kernel function. Inspired by this fact, we can define a simpler alternative governing operator that satisfies homogeneous Neumann BC

$$\mathcal{M}u(x) := cu(x) - \int_{\Omega} \widehat{C}(x-y) P_e u(y) dy \tag{13}$$

$$= cu(x) - \int_{\Omega} (1/2) \left(\widehat{C}(x-y) + \widehat{C}(x+y) \right) u(y)] dy. \tag{14}$$

In a similar fashion to (11) and (12), one can easily show that (14) satisfies the BC.

3 Perturbation Expansions

We construct approximations $\widetilde{\mathcal{L}}, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ to $\mathcal{L}, \mathcal{M}, \mathcal{N}$ using perturbation expansions in the form of Taylor polynomials by consistently keeping the size of the expansion neighborhood equal to δ in each case. This leads to Taylor polynomial of $u(y)$ at different y locations such as $y = x, -x, x-1, x+1$. That way, approximations of $u(y)$ all have error $\mathcal{O}(\delta^3)$, which means that we maintain consistent error among approximate operators.

For each operator, we have 3 intervals, on which the Taylor polynomials are guaranteed to have the size of the expansion neighborhood equal to δ in the corresponding y-range. We list the y-ranges, depict them in Figure 1 as shaded regions. Then, we utilize a Taylor polynomial which defines the approximate integrand $\widetilde{f}_{\cdot,\cdot}(x, y)$. Eventually, we calculate the approximate operator for the corresponding interval.

We easily see that $c = \int_{\Omega} \chi_{\delta}(y) dy = 2\delta$. For convenience, we prefer to use $\widetilde{\mathcal{L}}u(x) - 2\delta u(x)$, $\widetilde{\mathcal{M}}u(x) - 2\delta u(x)$, and $\widetilde{\mathcal{N}}u(x) - 2\delta u(x)$. The calculations for approximations $\widetilde{\mathcal{L}}, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ are given in a systematic way. We also report values at transition points for each approximate operator.

3.1 Operator $\widetilde{\mathcal{L}}$

The integrand is $f_{\mathcal{L}}(x, y) = -u(y)$. We have 3 intervals, left, center, and right denoted by ℓ, $\mathtt{c}$, and $\mathtt{r}$, respectively. $I_{\mathcal{L},\ell} := (-1, -1+\delta)$, $I_{\mathcal{L},\mathtt{c}} := (-1+\delta, 1-\delta)$, and $I_{\mathcal{L},\mathtt{r}} := (1-\delta, 1)$.

Operator $\widetilde{\mathcal{L}}_\ell$, $x \in I_{\mathcal{L},\ell} = (-1, -1+\delta)$

$$y \in R_{\mathcal{L},\ell} = (-1, x+\delta),\ \ y - x \in (-x-1, \delta) \subset (-\delta, \delta),\ \ |y-x| < \delta$$

$$\begin{aligned}
R_{\mathcal{L},\ell} : u(y) &= u(x) + (y-x)u'(x) + \frac{(y-x)^2}{2}u''(x) + \mathcal{O}(\delta^3),\\
\widetilde{f}_{\mathcal{L},\ell}(x,y) &= -u(x) - (y-x)u'(x) - \frac{(y-x)^2}{2}u''(x),\\
\widetilde{\mathcal{L}}_\ell u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{L},\ell}} \widetilde{f}_{\mathcal{L},\ell}(x,y)dy\\
&= [-x-1-\delta]u(x) + \frac{1}{2}\left[x-(-1-\delta)\right]\left[x-(-1+\delta)\right]u'(x)\\
&\quad + \frac{-1}{6}\left[x-(-1-\delta)\right]\left[x^2-(\delta-2)x+1-\delta+\delta^2\right]u''(x).
\end{aligned}$$

Operator $\widetilde{\mathcal{L}}_{\mathtt{c}}$, $x \in I_{\mathcal{L},\mathtt{c}} = (-1+\delta, 1-\delta)$

$$y \in R_{\mathcal{L},\mathtt{c}} = (x-\delta, x+\delta),\ \ y - x \in (-\delta, \delta),\ \ |y-x| < \delta$$

$$\begin{aligned}
\widetilde{f}_{\mathcal{L},\mathtt{c}}(x,y) &= \widetilde{f}_{\mathcal{L},\ell}(x,y),\\
\widetilde{\mathcal{L}}_{\mathtt{c}} u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{L},\mathtt{c}}} \widetilde{f}_{\mathcal{L},\mathtt{c}}(x,y)dy\\
&= \frac{-\delta^3}{3}u''(x) - 2\delta u(x).
\end{aligned}$$

Operator $\widetilde{\mathcal{L}}_{\mathtt{r}}$, $x \in I_{\mathcal{L},\mathtt{r}} = (1-\delta, 1)$

$$y \in R_{\mathcal{L},\mathtt{r}} = (x-\delta, 1),\ \ y - x \in (-\delta, -x+1) \subset (-\delta, \delta),\ \ |y-x| < \delta$$

$$\begin{aligned}
\widetilde{f}_{\mathcal{L},\mathtt{r}}(x,y) &= \widetilde{f}_{\mathcal{L},\ell}(x,y),\\
\widetilde{\mathcal{L}}_{\mathtt{r}} u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{L},\mathtt{r}}} \widetilde{f}_{\mathcal{L},\mathtt{r}}(x,y)dy\\
&= [x-1-\delta]u(x) + \frac{-1}{2}\left[x-(1+\delta)\right]\left[x-(1-\delta)\right]u'(x)\\
&\quad + \frac{1}{6}\left[x-(1+\delta)\right]\left[x^2+(\delta-2)x+1-\delta+\delta^2\right]u''(x).
\end{aligned}$$

Values of $\widetilde{\mathcal{L}}$ at boundary and transition points

$$
\begin{aligned}
\widetilde{\mathcal{L}}_\ell u(-1) \qquad\qquad &= \frac{-\delta^3}{6}u''(-1) + \frac{-\delta^2}{2}u'(-1) + \delta u(-1)\\
\widetilde{\mathcal{L}}_\ell u(-1+\delta) = \widetilde{\mathcal{L}}_{\mathtt{c}} u(-1+\delta) &= \frac{-\delta^3}{3}u''(-1+\delta)\\
\widetilde{\mathcal{L}}_{\mathtt{c}} u(1-\delta) = \widetilde{\mathcal{L}}_{\mathtt{r}} u(1-\delta) &= \frac{-\delta^3}{3}u''(1-\delta)\\
\widetilde{\mathcal{L}}_{\mathtt{r}} u(1) \qquad\qquad &= \frac{-\delta^3}{6}u''(1) + \frac{\delta^2}{2}u'(1) + \delta u(1).
\end{aligned}
$$

3.2 Operator $\widetilde{\mathcal{M}}$

The integrand is $f_{\mathcal{M}}(x,y) = -P_e u(y)$. Similar to the $\widetilde{\mathcal{L}}$ case, we have 3 intervals: $I_{\mathcal{M},\ell} := (-1,-1+\delta)$, $I_{\mathcal{M},\mathtt{c}} := (-1+\delta,1-\delta)$, and $I_{\mathcal{M},\mathtt{r}} := (1-\delta,1)$.

Operator $\widetilde{\mathcal{M}}_\ell$, $x \in I_{\mathcal{M},\ell} = (-1,-1+\delta)$

$$
\begin{aligned}
&y \in R_{\mathcal{M},\ell} = (x+2-\delta,1),\ \ y+x \in (2x+2-\delta,x+1) \subset (-\delta,\delta),\ \ |y+x| < \delta\\
&y \in R_{\mathcal{M},\ell-\mathtt{c}} = (-1,x+\delta),\ \ y-x \in (-x-1,\delta) \subset (-\delta,\delta),\qquad\quad |y-x| < \delta.
\end{aligned}
$$

$$
\begin{aligned}
R_{\mathcal{M},\ell} : u(y) &= u(-x) + (y+x)u'(-x) + \frac{(y+x)^2}{2}u''(-x) + \mathcal{O}(\delta^3),\\
R_{\mathcal{M},\ell-\mathtt{c}} : u(y) &= u(x) + (y-x)u'(x) + \frac{(y-x)^2}{2}u''(x) + \mathcal{O}(\delta^3),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{f}_{\mathcal{M},\ell}(x,y) &= -P_e u(x) + (y+x)P_o u'(x) - \frac{(y+x)^2}{2}P_e u''(x),\\
\widetilde{f}_{\mathcal{M},\ell-\mathtt{c}}(x,y) &= -P_e u(x) - (y-x)P_o u'(x) - \frac{(y-x)^2}{2}P_e u''(x).
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathcal{M}}_\ell u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{M},\ell}} \widetilde{f}_{\mathcal{M},\ell}(x,y)dy + \int\limits_{R_{\mathcal{M},\ell-\mathtt{c}}} \widetilde{f}_{\mathcal{M},\ell-\mathtt{c}}(x,y)dy\\
&= q_\ell(x)P_e u''(x) - (x+1-\delta)^2 P_o u'(x) - 2\delta P_e u(x). \qquad (15)
\end{aligned}
$$

Operator $\widetilde{\mathcal{M}}_{\mathtt{c}}$, $x \in I_{\mathcal{M},\mathtt{c}} = (-1+\delta,1-\delta)$

$$
y \in R_{\mathcal{M},\mathtt{c}} = (x-\delta,x+\delta),\ \ y-x \in (-\delta,\delta),\ \ |y-x| < \delta
$$

$$
\begin{aligned}
\widetilde{f}_{\mathcal{M},\mathtt{c}}(x,y) &= \widetilde{f}_{\mathcal{M},\ell-\mathtt{c}}(x,y),\\
\widetilde{\mathcal{M}}_{\mathtt{c}} u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{M},\mathtt{c}}} \widetilde{f}_{\mathcal{M},\mathtt{c}}(x,y)dy\\
&= \frac{-\delta^3}{3}P_e u''(x) - 2\delta P_e u(x).
\end{aligned}
$$

Operator $\widetilde{\mathcal{M}}_{\mathtt{r}},\ x \in I_{\mathcal{M},\mathtt{r}} = (1-\delta, 1)$

$$
\begin{array}{lll}
y \in R_{\mathcal{M},\mathtt{r}} = (-1, x-2+\delta), & y+x \in (x-1, 2x-2+\delta) \subset (-\delta,\delta), & |y+x| < \delta \\
y \in R_{\mathcal{M},\mathtt{r-c}} = (x-\delta, 1), & y-x \in (-\delta, -x+1) \subset (-\delta,\delta), & |y-x| < \delta.
\end{array}
$$

$$
\begin{aligned}
\widetilde{f}_{\mathcal{M},\mathtt{r}}(x,y) &= \widetilde{f}_{\mathcal{M},\ell}(x,y), \\
\widetilde{f}_{\mathcal{M},\mathtt{r-c}}(x,y) &= \widetilde{f}_{\mathcal{M},\ell-c}(x,y), \\
\widetilde{\mathcal{M}}_{\mathtt{r}} u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{M},\mathtt{r}}} \widetilde{f}_{\mathcal{M},\mathtt{r}}(x,y)dy + \int\limits_{R_{\mathcal{M},\mathtt{r-c}}} \widetilde{f}_{\mathcal{M},\mathtt{r-c}}(x,y)dy \\
&= q_{\mathtt{r}}(x) P_e u''(x) + (x-1+\delta)^2 P_o u'(x) - 2\delta P_e u(x). \qquad (16)
\end{aligned}
$$

Remark 2. Expressions of the coefficients $q_\ell(x)$ and $q_{\mathtt{r}}(x)$ of $P_e u''(x)$ in (15) and (16), respectively, are quite involved. So, we prefer not to report them.

Values of $\widetilde{\mathcal{M}}$ at boundary and transition points

$$
\begin{aligned}
\widetilde{\mathcal{M}}_\ell u(-1) - 2\delta u(-1) &= \frac{-\delta^3}{3} P_e u''(-1) - \delta^2 P_o u'(-1) - 2\delta P_e u(-1) \\
\widetilde{\mathcal{M}}_\ell u(-1+\delta) - 2\delta u(-1+\delta) &= \frac{-\delta^3}{3} P_e u''(-1+\delta) - 2\delta P_e u(-1+\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{c}} u(-1+\delta) - 2\delta u(-1+\delta) &= \frac{-\delta^3}{3} P_e u''(-1+\delta) - 2\delta P_e u(-1+\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{c}} u(1-\delta) - 2\delta u(1-\delta) &= \frac{-\delta^3}{3} P_e u''(1-\delta) - 2\delta P_e u(1-\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{r}} u(1-\delta) - 2\delta u(1-\delta) &= \frac{-\delta^3}{3} P_e u''(1-\delta) - 2\delta P_e u(1-\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{r}} u(1) - 2\delta u(1) &= \frac{-\delta^3}{3} P_e u''(1) + \delta^2 P_o u'(1) - 2\delta P_e u(1).
\end{aligned}
$$

3.3 Operator $\widetilde{\mathcal{N}}$

The integrand is $f_{\mathcal{N}}(x,y) = -P_e u(y)$. We have 3 intervals: $I_{\mathcal{N},\ell} := (-1,-\delta)$, $I_{\mathcal{N},\mathtt{c}} := (-\delta,\delta)$, and $I_{\mathcal{N},\mathtt{r}} := (\delta, 1)$.

Operator $\widetilde{\mathcal{N}}_\ell,\ x \in I_{\mathcal{N},\ell} = (-1,-\delta)$

$$
y \in R_{\mathcal{N},\ell} = (x+1-\delta, x+1+\delta),\ \ y-(x+1) \in (-\delta,\delta),\ \ |y-(x+1)| < \delta.
$$

$$R_{\mathcal{N},\ell} : u(y) = u(x+1) + [y-(x+1)]u'(x+1) + \frac{[y-(x+1)]^2}{2}u''(x+1) + \mathcal{O}(\delta^3),$$

$$\widetilde{f}_{\mathcal{N},\ell}(x,y) = -P_e u(x+1) - [y-(x+1)]P_o u'(x+1) - \frac{[y-(x+1)]^2}{2}P_e u''(x+1),$$

$$\begin{aligned}\widetilde{\mathcal{N}}_\ell u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{N},\ell}} \widetilde{f}_{\mathcal{N},\ell}(x,y)dy \\ &= \frac{-\delta^3}{3}P_e u''(x+1) - 2\delta P_e u(x+1).\end{aligned}$$

Operator $\widetilde{\mathcal{N}}_{\mathtt{c}}$, $x \in I_{\mathcal{N},\mathtt{c}} = (-\delta, \delta)$

$$\begin{aligned}&y \in R_{\mathcal{N},\ell-\mathtt{c}} = (x+1-\delta, 1), \quad y-(x+1) \in (-\delta, -x) \subset (-\delta,\delta), \quad |y-(x+1)| < \delta \\ &y \in R_{\mathcal{N},\mathtt{r}-\mathtt{c}} = (-1, x-1+\delta), \quad y-(x-1) \in (-x, \delta) \subset (-\delta,\delta), \quad |y-(x-1)| < \delta.\end{aligned}$$

$$R_{\mathcal{N},\ell-\mathtt{c}} : u(y) = u(x+1) + [y-(x+1)]u'(x+1) + \frac{[y-(x+1)]^2}{2}u''(x+1) + \mathcal{O}(\delta^3)$$

$$R_{\mathcal{N},\mathtt{r}-\mathtt{c}} : u(y) = u(x-1) + [y-(x-1)]u'(x-1) + \frac{[y-(x-1)]^2}{2}u''(x-1) + \mathcal{O}(\delta^3)$$

$$\begin{aligned}&\widetilde{f}_{\mathcal{N},\ell-\mathtt{c}}(x,y) = \widetilde{f}_{\mathcal{N},\ell}(x,y), \\ &\widetilde{f}_{\mathcal{N},\mathtt{r}-\mathtt{c}}(x,y) = -P_e u(x-1) - [y-(x-1)]P_o u'(x-1) - \frac{[y-(x-1)]^2}{2}P_e u''(x-1).\end{aligned}$$

$$\begin{aligned}&\widetilde{\mathcal{N}}_{\mathtt{c}} u(x) - 2\delta u(x) = \int\limits_{R_{\mathcal{N},\ell-\mathtt{c}}} \widetilde{f}_{\mathcal{N},\ell-\mathtt{c}}(x,y)dy + \int\limits_{R_{\mathcal{N},\mathtt{r}-\mathtt{c}}} \widetilde{f}_{\mathcal{N},\mathtt{r}-\mathtt{c}}(x,y)dy \\ &= \frac{1}{6}(x^3-\delta^3)P_e u''(x+1) - \frac{1}{6}(x^3+\delta^3)P_e u''(x-1) + \\ &\quad \frac{1}{2}(x^2-\delta^2)[-P_o u'(x+1) + P_o u'(x-1)] + (x-\delta)P_e u(x+1) - (x+\delta)P_e u(x-1).\end{aligned}$$

Operator $\widetilde{\mathcal{N}}_{\mathtt{r}}$, $x \in I_{\mathcal{N},\mathtt{r}} = (\delta, 1)$

$$y \in R_{\mathcal{N},\mathtt{r}} = (x-1-\delta, x-1+\delta), \quad y-(x-1) \in (-\delta,\delta), \quad |y-(x-1)| < \delta.$$

$$\begin{aligned}\widetilde{f}_{\mathcal{N},\mathtt{r}}(x,y) &= \widetilde{f}_{\mathcal{N},\mathtt{r}-\mathtt{c}}(x,y), \\ \widetilde{\mathcal{N}}_{\mathtt{r}} u(x) - 2\delta u(x) &= \int\limits_{R_{\mathcal{N},\mathtt{r}}} \widetilde{f}_{\mathcal{N},\mathtt{r}}(x,y)dy \\ &= \frac{-\delta^3}{3}P_e u''(x-1) - 2\delta P_e u(x-1).\end{aligned}$$

Values of $\widetilde{\mathcal{N}}$ at boundary and transition points

$$\begin{aligned}
\widetilde{\mathcal{N}}_{\ell}u(-1) - 2\delta u(-1) &= \frac{-\delta^3}{3} P_e u''(0) - 2\delta P_e u(0) \\
\widetilde{\mathcal{N}}_{\ell}u(-\delta) - 2\delta u(-\delta) &= \frac{-\delta^3}{3} P_e u''(1-\delta) - 2\delta P_e u(1-\delta) \\
\widetilde{\mathcal{N}}_{\mathtt{c}}u(-\delta) - 2\delta u(-\delta) &= \frac{-\delta^3}{3} P_e u''(1-\delta) - 2\delta P_e u(1-\delta) \\
\widetilde{\mathcal{N}}_{\mathtt{c}}u(\delta) - 2\delta u(\delta) &= \frac{-\delta^3}{3} P_e u''(\delta-1) - 2\delta P_e u(\delta-1) \\
\widetilde{\mathcal{N}}_{\mathtt{r}}u(\delta) - 2\delta u(\delta) &= \frac{-\delta^3}{3} P_e u''(\delta-1) - 2\delta P_e u(\delta-1) \\
\widetilde{\mathcal{N}}_{\mathtt{r}}u(1) - 2\delta u(1) &= \frac{-\delta^3}{3} P_e u''(0) - 2\delta P_e u(0).
\end{aligned}$$

4 Comparison of Operators

4.1 Comparison in the Bulk

The interval $(-1+\delta, 1-\delta)$ is usually referred as the bulk of the domain. The behavior in the bulk is considered to be the main behavior of the operator especially when $\delta \ll 1$. That is why, it is important to find out the operator behavior in the bulk. By construction, the notion of bulk is slightly different for the $\mathcal{N}$ operator. The intervals $(-1, -\delta)$ and $(\delta, 1)$ will be referred as bulk in the case of $\mathcal{N}$. We list the bulk behavior of each operator:

$$\widetilde{\mathcal{L}}_{\mathtt{c}}u(x) = \frac{-\delta^3}{3} u''(x), \quad x \in (-1+\delta, 1-\delta), \tag{17}$$

$$\widetilde{\mathcal{M}}_{\mathtt{c}}u(x) - 2\delta u(x) = \frac{-\delta^3}{3} P_e u''(x) - 2\delta P_e u(x), \quad x \in (-1+\delta, 1-\delta), \tag{18}$$

$$\widetilde{\mathcal{N}}_{\ell}u(x) - 2\delta u(x) = \frac{-\delta^3}{3} P_e u''(x+1) - 2\delta P_e u(x+1), \quad x \in (-1, -\delta), \tag{19}$$

$$\widetilde{\mathcal{N}}_{\mathtt{r}}u(x) - 2\delta u(x) = \frac{-\delta^3}{3} P_e u''(x-1) - 2\delta P_e u(x-1), \quad x \in (\delta, 1). \tag{20}$$

We start comparing the operators with $\widetilde{\mathcal{L}}_{\mathtt{c}}$ and $\widetilde{\mathcal{M}}_{\mathtt{c}}$. Then, by substituting $u = P_e u$ in (18) and using $P_e^2 = P_e$, we arrive at

$$\widetilde{\mathcal{M}}_{\mathtt{c}} P_e u(x) = \frac{-\delta^3}{3} P_e u''(x). \tag{21}$$

In order to match (17) with (21), we also substitute $u = P_e u$ and we get

$$\widetilde{\mathcal{L}}_{\mathtt{c}} P_e u(x) = \frac{-\delta^3}{3} P_e u''(x).$$

Then, we conclude that the action of $\widetilde{\mathcal{L}}_{\mathtt{c}}$ and $\widetilde{\mathcal{M}}_{\mathtt{c}}$ agree in the bulk when restricted to the even component of $u(x)$.

In order to compare $\widetilde{\mathcal{N}}_{\ell}$ and $\widetilde{\mathcal{N}}_{\mathtt{r}}$ with $\widetilde{\mathcal{L}}_{\mathtt{c}}$, we substitute $u = P_e u$ in (19) and (20), which gives us the following results:

$$\widetilde{\mathcal{N}}_{\ell} P_e u(x) - 2\delta P_e u(x) = \frac{-\delta^3}{3} P_e u''(x+1) - 2\delta P_e u(x+1) \tag{22}$$

$$\widetilde{\mathcal{N}}_{\mathtt{r}} P_e u(x) - 2\delta P_e u(x) = \frac{-\delta^3}{3} P_e u''(x-1) - 2\delta P_e u(x-1). \tag{23}$$

In order to cancel the $2\delta P_e u(x)$ with $2\delta P_e u(x+1)$ and $2\delta P_e u(x-1)$ in (22) and (23), respectively, we need to make the following assumption:

$$u(x) = u(x-1) = u(x+1), \quad x \in (-1, -\delta) \cup (\delta, 1). \tag{24}$$

This property holds, for instance, when u is 1-periodic. Namely,

$$u(x) = u(x-1), \quad x \in \mathbb{R}. \tag{25}$$

We may conclude that the assumption (24) is triggered because of the half-wave symmetry property, noted in Remark 1, employed when constructing the integral operator $\mathcal{N}$. In summary, we conclude that $\widetilde{\mathcal{L}}_{\mathtt{c}}$ agrees with $\widetilde{\mathcal{N}}_{\ell}$ and $\widetilde{\mathcal{N}}_{\mathtt{r}}$ when restricted to the even component of $u(x)$ where $u(x)$ is 1-periodic.

4.2 Comparison of Higher Order Approximations in the Bulk

If we use a higher order Taylor approximation, for instance, for the $\widetilde{\mathcal{L}}_{\ell}$ operator $x \in I_{\mathcal{L},\ell} = (-1, -1+\delta)$, we get following expansion of $y \in R_{\mathcal{L},\ell} = (-1, x+\delta)$

$$u(y) = \left(I + (y-x)D + \cdots + \frac{(y-x)^{2n}}{(2n)!} D^{2n} \right) u(x) + \mathcal{O}(\delta^{2n+1}).$$

Then the error of the following operators is $\mathcal{O}(\delta^{2n+2})$.

$$\begin{aligned}
\widetilde{\mathcal{L}}_{\mathtt{c}} u(x) &= (-2) \left(\frac{\delta^3}{3!} D^2 + \frac{\delta^5}{5!} D^4 + \cdots + \frac{\delta^{2n+1}}{(2n+1)!} D^{2n} \right) u(x), \\
\widetilde{\mathcal{M}}_{\mathtt{c}} P_e u(x) &= (-2) \left(\frac{\delta^3}{3!} D^2 + \frac{\delta^5}{5!} D^4 + \cdots + \frac{\delta^{2n+1}}{(2n+1)!} D^{2n} \right) P_e u(x), \\
\widetilde{\mathcal{N}}_{\ell} P_e u(x) &= (-2) \left(\frac{\delta^3}{3!} D^2 + \frac{\delta^5}{5!} D^4 + \cdots + \frac{\delta^{2n+1}}{(2n+1)!} D^{2n} \right) P_e u(x+1), \\
\widetilde{\mathcal{N}}_{\mathtt{r}} P_e u(x) &= (-2) \left(\frac{\delta^3}{3!} D^2 + \frac{\delta^5}{5!} D^4 + \cdots + \frac{\delta^{2n+1}}{(2n+1)!} D^{2n} \right) P_e u(x-1).
\end{aligned}$$

Note that all these expressions on the right hand side can be written as a function of D^2. This is an indication that all the above approximate operators are functions of the Laplace; see the extended discussion in [7].

4.3 Comparison at the Boundary and Transition Points

We monitor where we can capture the factor $\frac{-\delta^3}{3}$ next to $u''(x)$ and $P_e u''(x)$ terms. We consider this as an indication that the bulk behavior is captured at that point. By using transition values computed in Section 3.1, first note that

$$\begin{aligned}
\widetilde{\mathcal{L}}_\ell u(-1+\delta) &= \widetilde{\mathcal{L}}_{\mathtt{c}} u(-1+\delta) = \frac{-\delta^3}{3} u''(-1+\delta) \\
\widetilde{\mathcal{L}}_{\mathtt{r}} u(1-\delta) &= \widetilde{\mathcal{L}}_{\mathtt{c}} u(1-\delta) \quad = \frac{-\delta^3}{3} u''(1-\delta).
\end{aligned}$$

At transition points $x = -1+\delta, 1-\delta$, we conclude that we can define a continuous extension of $\widetilde{\mathcal{L}}$ from the pieces $\widetilde{\mathcal{L}}_\ell, \widetilde{\mathcal{L}}_{\mathtt{c}}$, and $\widetilde{\mathcal{L}}_{\mathtt{r}}$.

In order to monitor boundary and bulk behavior, we need to manipulate boundary and transition expressions of the $\widetilde{\mathcal{M}}$ given in Section 3.2 by $u = P_e u$. Then, using implications of (2), i.e., $P_e P_o = 0$ and $P_e^2 = P_e$, we obtain

$$\begin{aligned}
\widetilde{\mathcal{M}}_\ell P_e u(-1) &= \frac{-\delta^3}{3} P_e u''(-1) \\
\widetilde{\mathcal{M}}_\ell P_e u(-1+\delta) &= \frac{-\delta^3}{3} P_e u''(-1+\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{c}} P_e u(-1+\delta) &= \frac{-\delta^3}{3} P_e u''(-1+\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{r}} P_e u(1-\delta) &= \frac{-\delta^3}{3} P_e u''(1-\delta) \\
\widetilde{\mathcal{M}}_{\mathtt{r}} P_e u(1) &= \frac{-\delta^3}{3} P_e u''(1).
\end{aligned}$$

Similar to the $\widetilde{\mathcal{L}}$ case, we can define a continuous extension of $\widetilde{\mathcal{M}} P_e$ at transition points from the pieces $\widetilde{\mathcal{M}}_\ell P_e, \widetilde{\mathcal{M}}_{\mathtt{c}} P_e$, and $\widetilde{\mathcal{M}}_{\mathtt{r}} P_e$.

In order to monitor boundary and bulk behavior of $\widetilde{\mathcal{N}}$, we manipulate boundary and transition expressions given in Section 3.3 by $u = P_e u$. Then, by assuming (25), we obtain

$$\begin{aligned}
\widetilde{\mathcal{N}}_\ell P_e u(-1) &= \frac{-\delta^3}{3} P_e u''(-1) \\
\widetilde{\mathcal{N}}_\ell P_e u(-\delta) = \widetilde{\mathcal{N}}_{\mathtt{c}} P_e u(-\delta) &= \frac{-\delta^3}{3} P_e u''(-\delta) \\
\widetilde{\mathcal{N}}_{\mathtt{c}} P_e u(\delta) = \widetilde{\mathcal{N}}_{\mathtt{r}} P_e u(\delta) \quad &= \frac{-\delta^3}{3} P_e u''(\delta) \\
\widetilde{\mathcal{N}}_{\mathtt{r}} P_e u(1) \quad &= \frac{-\delta^3}{3} P_e u''(1).
\end{aligned}$$

We can also define a continuous extension of $\widetilde{\mathcal{N}} P_e$ at transition points from the pieces $\widetilde{\mathcal{N}}_\ell P_e$, $\widetilde{\mathcal{N}}_{\mathtt{c}} P_e$, and $\widetilde{\mathcal{N}}_{\mathtt{r}} P_e$.

Note that values of $\widetilde{\mathcal{M}}P_e$ at boundary and bulk points exhibit the bulk behavior of $\widetilde{\mathcal{L}}$. In addition, under assumption (25), values of $\widetilde{\mathcal{N}}_{\mathbf{r}}P_e$ and $\widetilde{\mathcal{N}}_\ell P_e$ at boundary points also exhibit the bulk behavior of $\widetilde{\mathcal{L}}$. These might be indications that the surface effect issue observed in PD can be eliminated if $\mathcal{M}$ and $\mathcal{N}$ are used as governing operators. This is a future research avenue.

5 Conclusion

The important property we seek is to obtain $-\delta^3/3$ as the coefficient of the term with the second derivative. We identify this as the bulk behavior. Both $\widetilde{\mathcal{M}}P_e$ and $\widetilde{\mathcal{N}}P_e$ exhibit the same bulk behavior as $\widetilde{\mathcal{L}}P_e$. Furthermore, the bulk behavior is also observed at all boundary and transition points for $\widetilde{\mathcal{M}}P_e$ and $\widetilde{\mathcal{N}}P_e$. The comparison of $\widetilde{\mathcal{N}}$ to $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{M}}$ requires the assumption of (25). Due to this restriction, we conclude that $\widetilde{\mathcal{L}}$ agrees with $\widetilde{\mathcal{M}}$ more than it does with $\widetilde{\mathcal{N}}$.

In the expansion of $\widetilde{\mathcal{M}}P_e$, the coefficients of $P_e u''(x)$ are all equal to $-\delta^3/3$ at transition points as well as at boundary points. This can be interpreted as the best possible agreement with the Laplace operator. Such an agreement may indicate that the surface effects observed in PD can be eliminated especially when $\mathcal{M}$ is used as governing operator. For future research, by eliminating the assumptions $u = P_e u$ and (25), we plan to construct governing operators that agree with $\mathcal{L}$ in the bulk.

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